Foundations of the Transfer Principle in Nonstandard Analysis

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Outline

Introduction

Motivation

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Nonstandard Analysis is fundamentally about a reformulation of mathematics itself to accommodate infinitesimal elements in a way that does not violate any natural intuition concerning how numbers work. Nonstandard analysis finds itself applications in unexpected places, such as Ramsey theory, where mathematicians work with infinitesimals and use the transfer principle to justify that whatever work they have done is also true in the regular reals.

Notational Conventions

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- $\mathcal{P}(X)$ is called the "powerset" of X. It is the set of all subsets of X.
- For any set X, ${}^*X$ denotes the hyper-extension (or nonstandard extension) of X. For example, ${}^*\mathbb{R}$ denotes the hyperreals.

The Hyperreals

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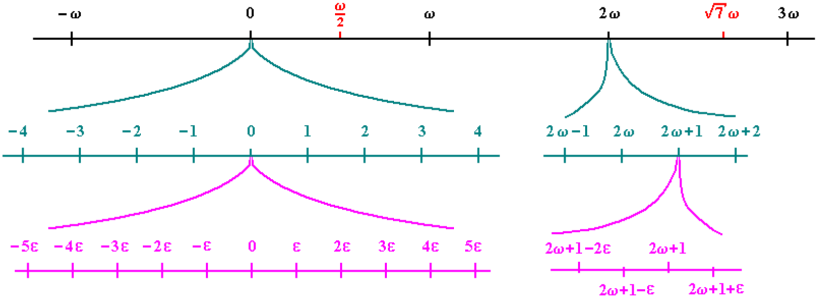

Figure: Figure 1. Hyperreal Structure (Schmidtke)

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It should be noted that $\mathbb{R}$ actually admits an infinitesimal element, that being 0.

Infinite/Unlimited Elements

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Definition (Infinite Elements)

We call δ an *infinite* element if $|\delta| > n$ for all $n \in \mathbb{N}$.

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One can also define $^*\mathbb{Z}$ and $^*\mathbb{Q}$.

Transfer Principle

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The hyperreals obey the *transfer principle*. This principle informally says for the hyperreals that "statements are true in the hyperreals if and only if they are true in the reals". However, we aim to find a more general version.

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The star maps $X \mapsto {}^*X$. Notice that the star map's definition has the *transfer principle* built into it.

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   Non-Example: "This sentence is a lie."
2. First order bounded: Every quantifier $(\forall, \exists)$ is bounded by a set.
   Example: $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x + y = 0)$.
   Non-Example: $\forall x \forall y \subseteq Y\ (\dots)$.

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This transfer principle applies to many structures such as $\mathbb{R}$ and ${}^*\mathbb{R}$, but also $\mathbb{N}$ and ${}^*\mathbb{N}$.

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We naturally expect $\infty = \infty + 1$.
However, the transfer principle tells us that "for all $n \in {}^*\mathbb{N}$, $n < n + 1$," must be true. So, the infinite elements in ${}^*\mathbb{N}$ still behave like regular numbers.

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1. Formulate P as a statement-type that'll work by the transfer principle.
2. Analyze and prove P in the nonstandard extension ${}^*X$. We are free to take advantage of the nonstandard elements and behavior that ${}^*X$ might have.
3. Because P is true in ${}^*X$, it must also be true in X which completes the proof.

Failure of Transfer on Subsets

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With respect to the transfer principle: quantifying $\forall x \subseteq X\ (\dots)$ is not the same as $\forall x \in \mathcal{P}(X)\ (\dots)$ because ${}^*\mathcal{P}(X) \neq \mathcal{P}({}^*X)$ in general.

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Once again, consider the structure of ${}^*\mathbb{N}$:

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It is true that "every nonempty subset of $\mathbb{N}$ has a least element." However, it is clear that the set ${}^*\mathbb{N} \setminus \mathbb{N} = \{\ldots, N-1, N, N+1, \ldots\}$ does not have this property. This is not a violation of the transfer principle. This instead tells us that ${}^*\mathbb{N}$ does not believe ${}^*\mathbb{N} \setminus \mathbb{N}$ to be a nonempty subset of itself (so it is external).

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Subsets that ${}^*\mathbb{N}$ believes to exist is called internal.

Ultrapower Model

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We want to define an actual star map that satisfies the transfer principle to show that we are studying a well-defined object. We do this by virtue of the *ultrapower model*, which is a very strong set theoretic object.

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$$\begin{aligned}\sigma =& \langle 1, \mathbf{1}, \mathbf{1}, \ldots \rangle \\ \tau =& \langle 4, \mathbf{1}, \mathbf{1}, \ldots \rangle.\end{aligned}$$

In other words, $\sigma(i) = \tau(i)$ for $i > 1$. These two sequences are equal "almost everywhere," so $\sigma \equiv_{\mathcal{U}} \tau$ in ${}^*\mathbb{R}$. σ would represent the number 1 in ${}^*\mathbb{R}$.

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Proof Sketch

For each $r \in \mathbb{R}$, we represent r in the hyperreals with the sequence $\langle r, r, r, \ldots \rangle$. The positions where $N(i) > n$ for some $n \in \mathbb{N}$ occur "almost everywhere" (they form a cofinite set). This implies that $N > n$ for all $n \in \mathbb{N}$, concluding that N is indeed an infinite element.

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This tells us that the ultrapower model actually does work as a star map, and the more general theorem tells us that similar constructions can work for more general structures beyond just $\mathbb{R}$ or $\mathbb{N}$.