Infinitary Combinatorics

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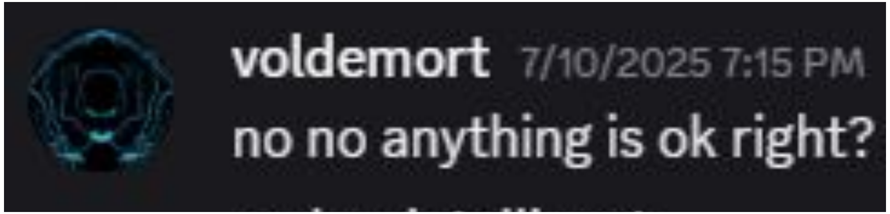
Acknowledgements

- Paul Yoo (Officer at Thomas Jefferson's Varsity Math Team)



Motivation

The study of infinitary combinatorics will not be particularly useful in any high school math olympiad or competition, ever.

You can either perceive this lecture as a break from that form of mathematics and an exploration into higher level pure math, or you can see it as an exercise in developing your proof skills*.

*Particularly, such proofs covered in this lecture are of the same nature you will see in power rounds, such as the PUMaC Power Round. However, there is no guarantee that this specific topic will ever be covered in one of those competitions.

Introduction

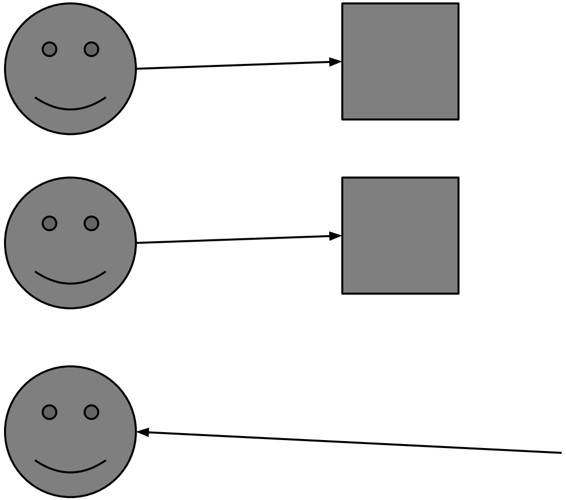
Infinitary combinatorics, or combinatorial set theory, is a field of mathematics that extends finite combinatorial results to the infinite. Of particular interest in the field is extending Ramsey Theory, which covers generalized pigeonhole principles. You do not need to know finite Ramsey Theory for this lecture.

We will cover a singular important result and go over the proof.

Background

Pigeonhole Principle

If you have n objects and m containers with n > m, then at least one container will have more than 1 object.



This guy must go into one of the already taken containers, which means one container has more than one person.

Proof

Let $n > m$. We will assign each object a container.

Suppose for contradiction that each container has at most one object. Then there are only at most $n = m$ objects, contradiction. :)

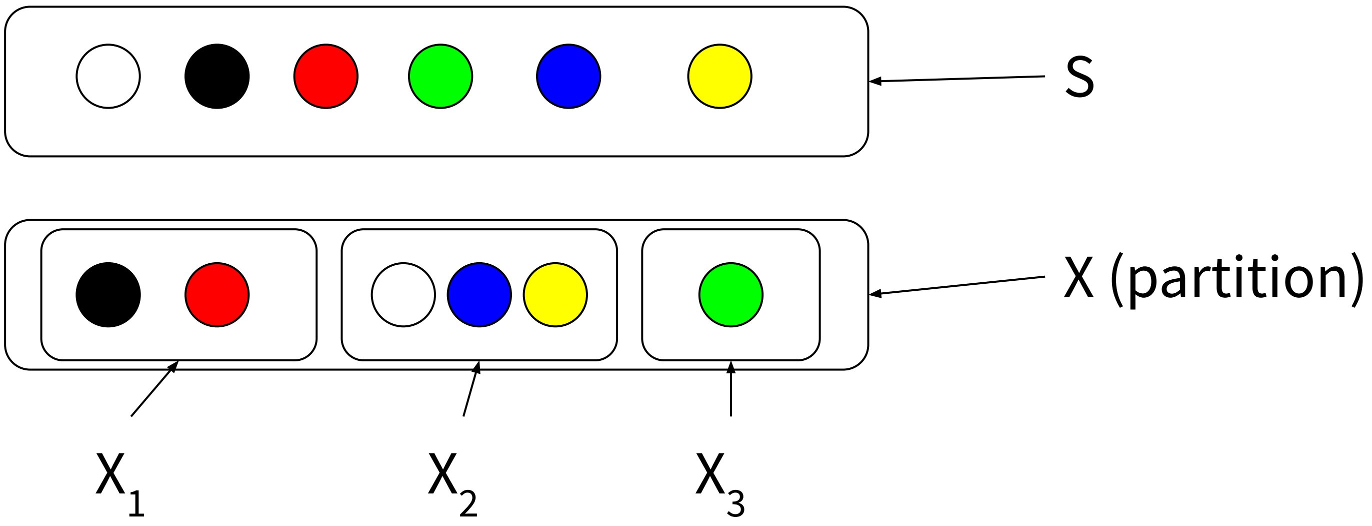
Set Theoretic Knowledge

Here are some facts to know:

- $|A|$ means the number of elements in A
- $\omega = \{0,1,2,3,\ldots\}$ (think of this as the natural numbers)
  - $|\omega|$ is infinite
  - You can biject many sets into this set, proving they have the same size. For example, the set of nonnegative even numbers. The bijective function is $f(x) = 2x$.
- $[A]^n$ is the set of all subsets of A that have size n.
  - If $A = \{0,1,2\}$, then $[A]^2 = \{\{0,1\}, \{0,2\}, \{1,2\}\}$
  - Basically “A choose n”!

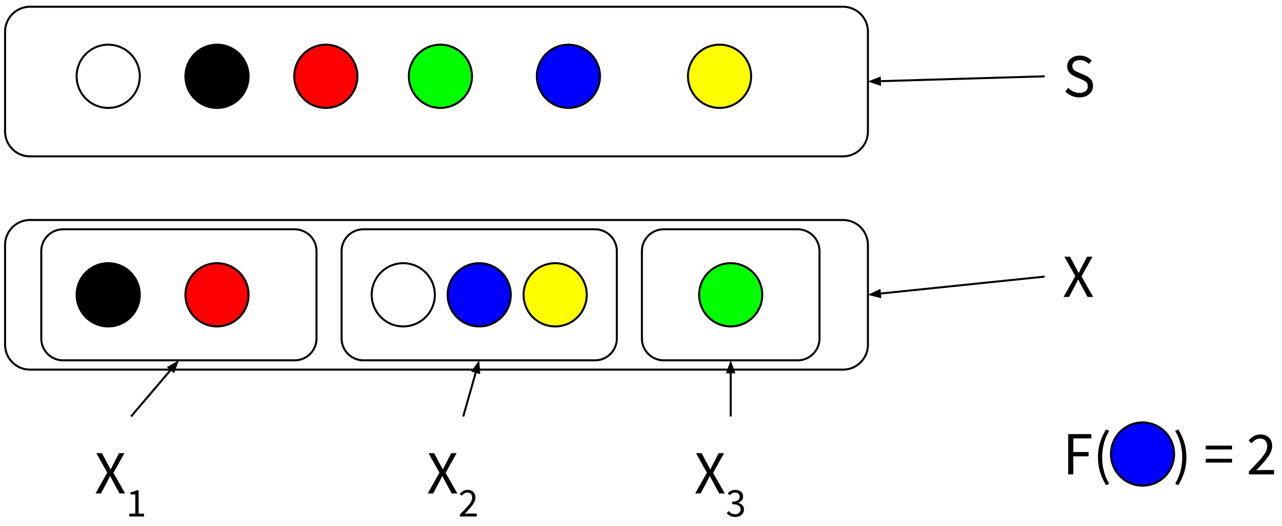
Set Theoretic Knowledge (Part 2)

A partition of a set S is a family of sets X where elements in X are disjoint and the union of all elements in X is S.



Set Theoretic Knowledge (Part 3)

We can have functions F : S -> {1,...,k} to describe a partition with k partitions. Basically: suppose we want to partition S into k pieces. We make $X_1,\ldots,X_k$ as the partition. If $s \in S$ and it is in partition X_i, then F(s) = i.



Set Theoretic Knowledge (Part 4)

Suppose X is a partition of $[A]^n$. A set $H \subseteq A$ is *homogenous* if every element of $[H]^n$ is included in the same partition. In other words, every n element subset of H is in the same partition.

$A = \{0, 1, 2\}$

$[A]^2 = \{\{0,1\}, \{0,2\}, \{1,2\}\}$

$X = \{$

$X_1 \rightarrow \{\{0,1\}, \{0,2\}\},$

$X_2 \rightarrow \{\{1,2\}\}$

$\}$

$H = \{1, 2\}$

$[H]^2 = \{\{1, 2\}\}$

Everything in $[H]^2$ is in the same partition so H is homogenous for this X (in this case n=2).

Combinatorial Principles

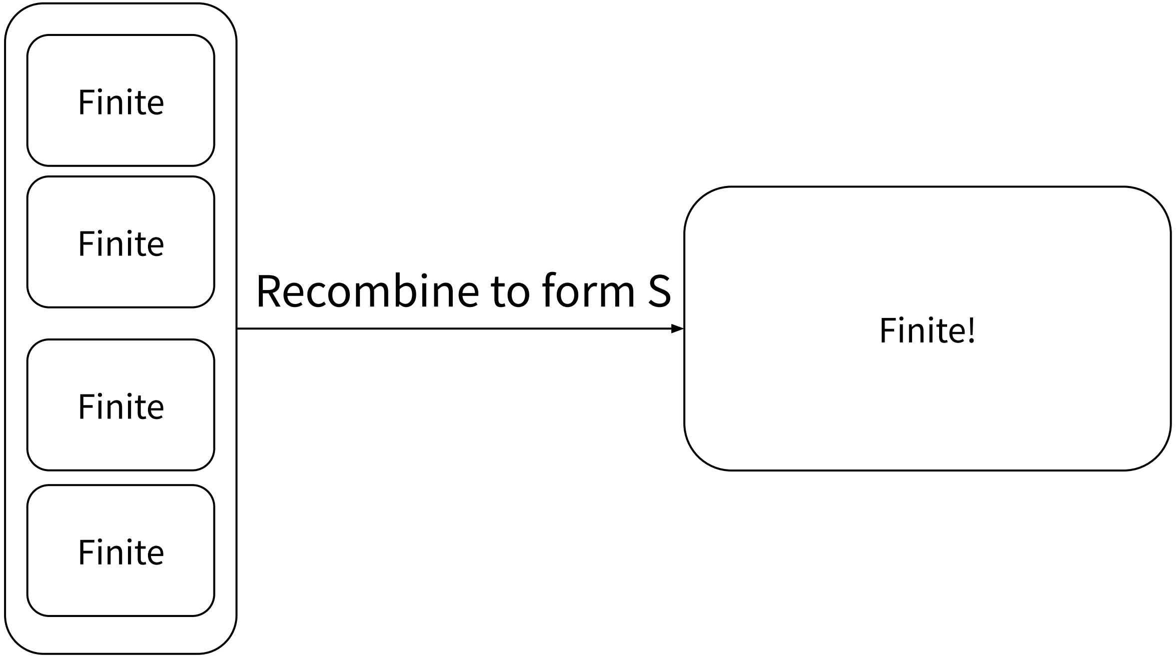
Infinite Pigeonhole Principle

If an infinite set is partitioned into finitely many pieces, then at least one piece is infinite.

(try to prove this yourself)

Proof

Suppose X is our partition of the infinite set S. If every piece of X is finite, then their union is finite. Thereby S is finite, contradiction. :)



Ramsey Theorem

For positive integers n, k, every partition $\{X_1,\ldots,X_k\}$ of $[\omega]^n$ has an infinite homogeneous set.

In other words, for every $F : [\omega]^n \to \{1,\ldots,k\}$ there is an infinite subset $H \subseteq \omega$ such that F is constant on $[H]^n$. The function F being constant on $[H]^n$ means that for any two elements x, y in $[H]^n$, $F(x) = F(y)$.

(the second statement is simply the first expanded and using the function version of a partition)

What did that even mean?

Basically, we want to prove a version of the infinite pigeonhole principle but for $[\omega]^n$ and $[H]^n$ instead of just ω.

We want to prove that for a partition of the n element subsets of ω, there an infinite subset of ω such that its n element subsets are all in the same partition.

Proof (n = 1 case)

We will induct over n.

For the base case, n = 1, we have a very neat insight.

Hint: $[\omega]^1 = \{\{0\}, \{1\}, \{2\}, \ldots\}$ and $\omega = \{0, 1, 2, \ldots\}$

Proof (n = 1 solution)

We can easily biject $[\omega]^1$ into ω and **preserve structure**. At this point, it is simply just the aforementioned infinite pigeonhole principle!

Think about it, we have an infinite set (ω), and we need to find an infinite set H where all 1-element subsets of H are in the same partition (in other words, every element of H is in the same partition, hence an infinite partition.)

This won't work on higher cases since a bijection from $[\omega]^{>1}$ to ω doesn't necessarily preserve structure. Unfortunate.

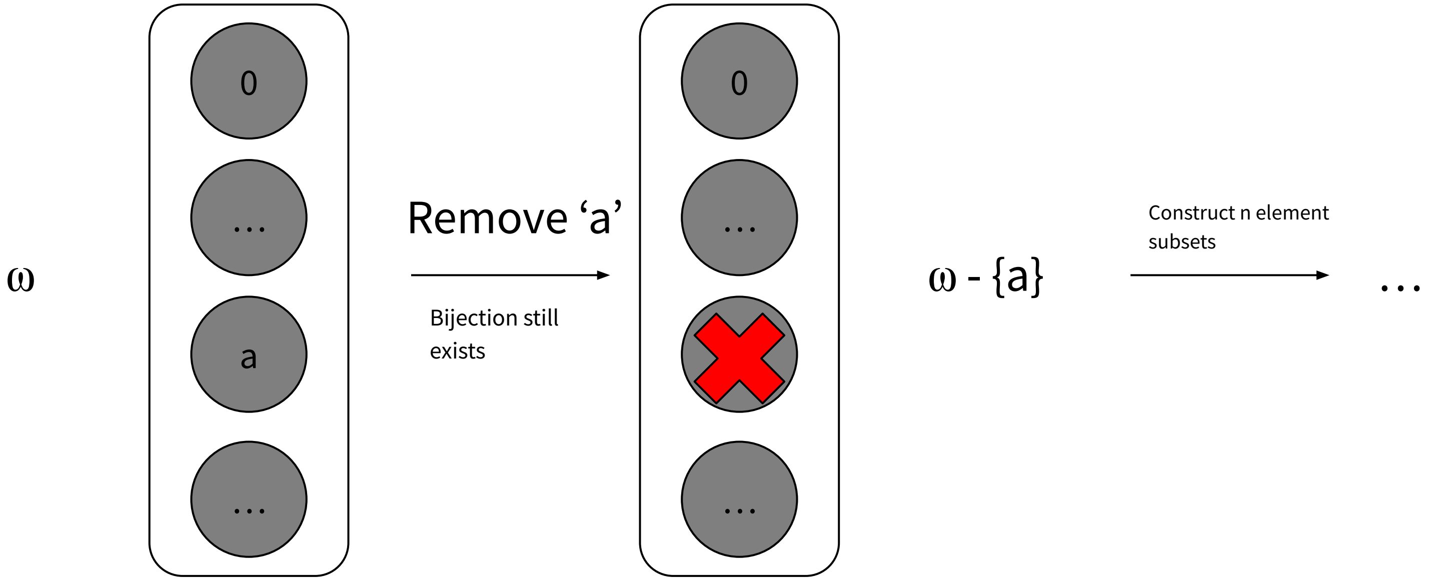
Proof Idea for Inductive Case

Let's go over the general idea first. The whole point of an inductive proof is to use the fact that it is true for n to prove it is true for n+1. This could work great on the function F which sends $[\omega]^n$ to {1,...,k}. It's equivalent and easy to work with. Therefore, somehow, we need to show that if it is true for all $F : [\omega]^n \to \{1,...,k\}$, then it is true for all $F : [\omega]^{n+1} \to \{1,...,k\}$.

Trying to Construct that Function

So, we need to show that every $F : [\omega]^{n+1} \rightarrow \{1,\ldots,k\}$ is constant on some $[H]^{n+1}$ where H is an infinite subset of ω.

Let us construct a family of functions $F_a : [\omega - \{a\}]^n \rightarrow \{1,\ldots,k\}$



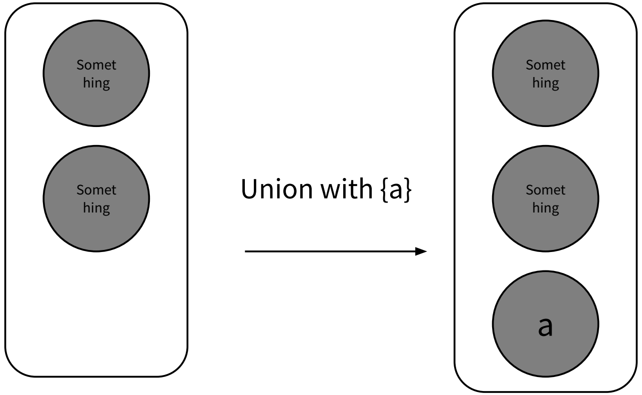
Trying to Construct that Function (Part 2)

For each $F_a : [\omega - \{a\}]^n \rightarrow \{1,\ldots,k\}$: $F_a(x) = F(\{a\} \cup x)$

So: $F_1(\{2,3\}) = F(\{1,2,3\})$ (in this case n=2, n+1=3)

We do this so we can try to abuse the property that it works for n onto n+1.

While the inductive hypothesis itself states it's true for $[\omega]^n$, we can easily prove it is actually true for any $[S]^n$ where S is an infinite set that bijects into ω.



Continued Proof - A Sequence

From the induction hypothesis we know that for all $a \in \omega$ and $S \subseteq \omega$, there exists an $H^S_a \subseteq S - \{a\}$ where F_a is constant over $[H^S_a]^n$ (use the fact from the previous slide).

Let us construct an infinite sequence $a_0, a_1, \ldots$ and $S_0, S_1, \ldots$

$a_0 = 0$

$S_0 = \omega$

At this point in the proof, you may get lost so I will try my best to help visualize.

Proof Sequence (Part 2)

We will decide to build our sequence as so (see bottom right). Recall that $H^S_a \subseteq S - \{a\}$.

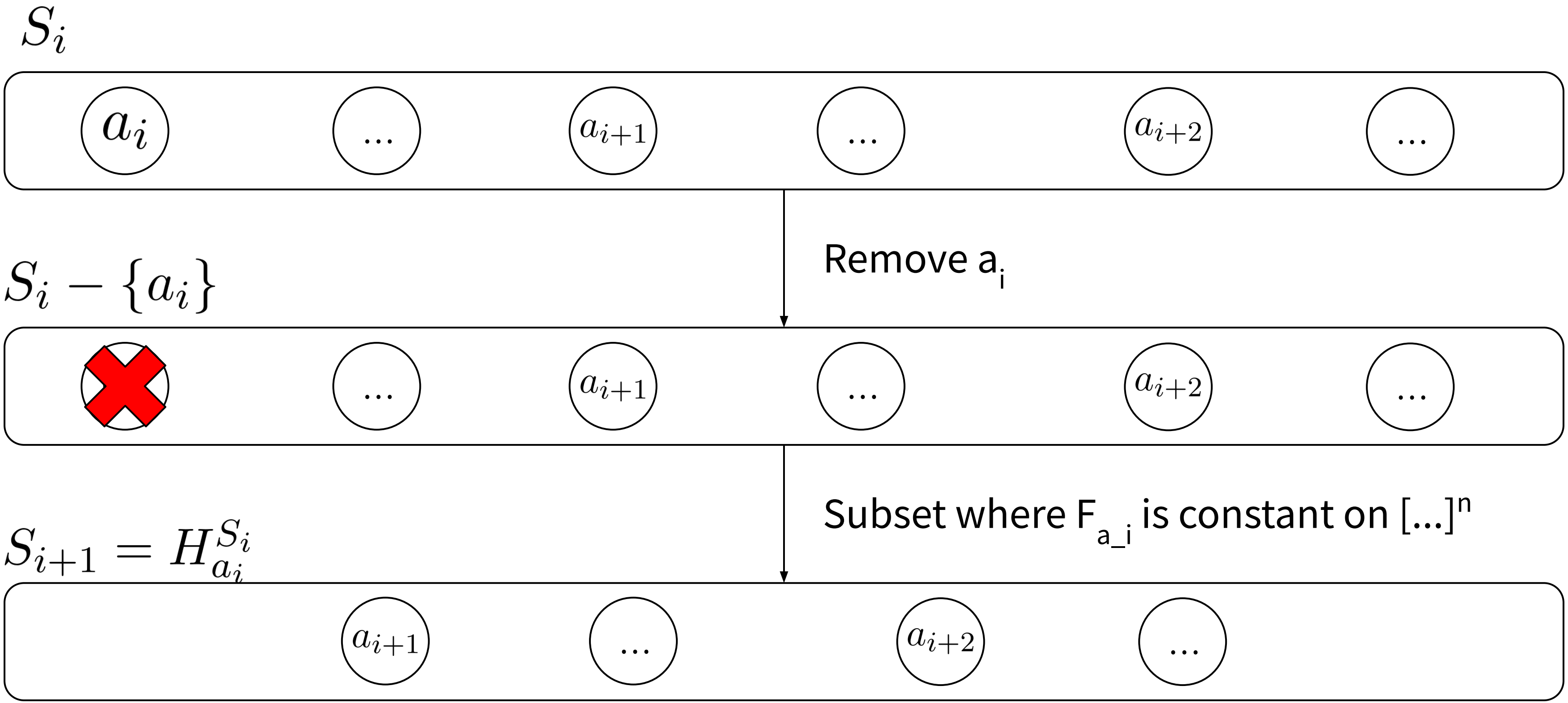
If you think about how every S_{i+1} works, it is actually is just $H^{S\_i}_{a\_i} \subseteq S_i - \{a_i\}$ such that F_{a_i} is constant on $[H^{S\_i}_{a\_i}]^n$. So F_{a_i} is constant on $[S_{i+1}]^n$. Additionally, the set $\{a_j : j > i\}$ is a subset of S_{i+1} (check the definition of a_{i+1} to verify). Since F_{a_i} is constant on $[S_{i+1}]^n$, it will be constant on $[\{a_j : j > i\}]^n$ as it is a subset. Visualization on next slide.

$$S_{i+1} = H^{S_i}_{a_i}$$

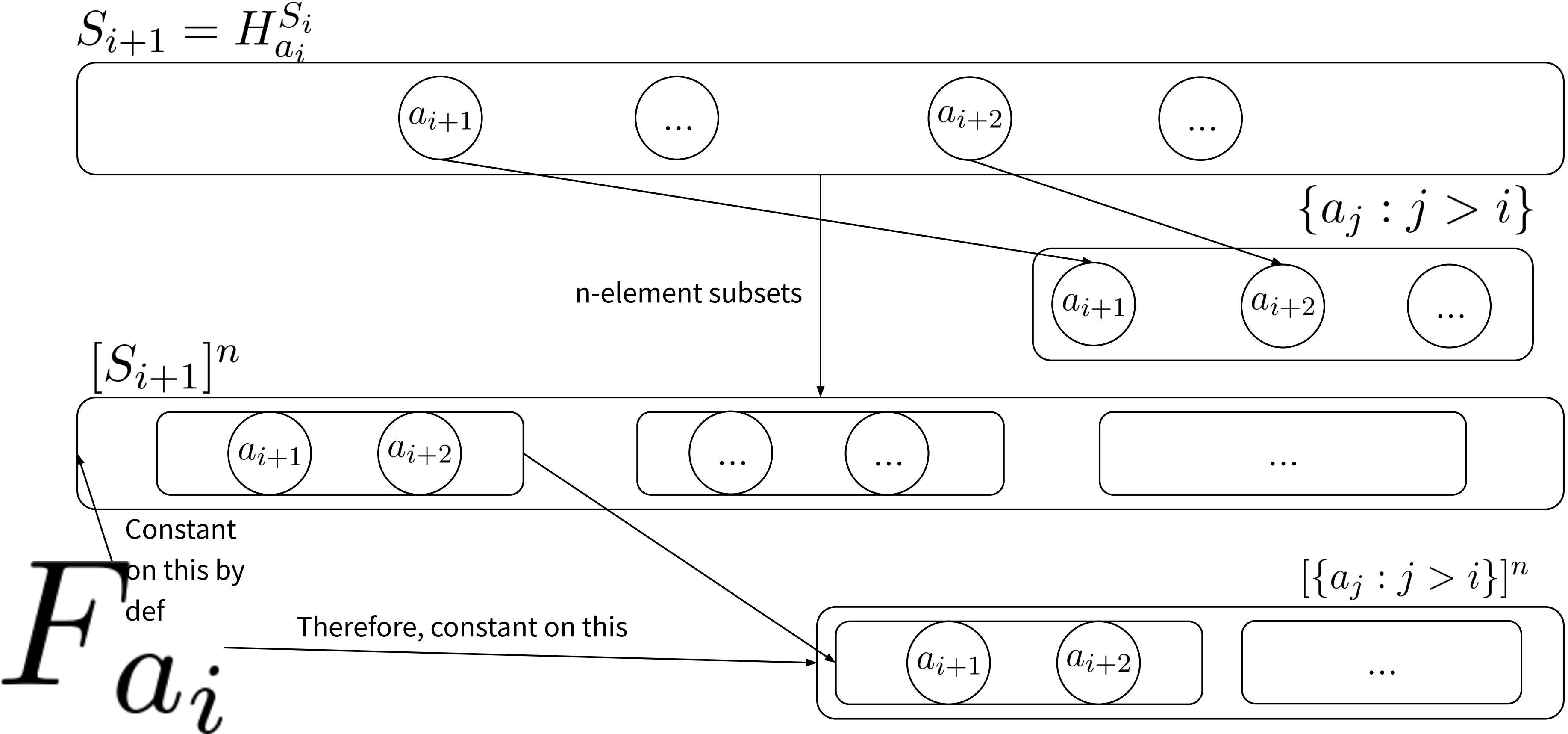
$$a_{i+1} = \text{the least element of } S_{i+1} \text{ greater than } a_i$$

$a_0 = 0, S_0 = \omega$

Proof Sequence Visualization



Proof Sequence Visualization (Part 2)



Continuation of Proof

We've established F_{a_i} to be constant on $[\{a_j : j > i\}]^n$. In other words, $F_{a\_i}(\{a_{i+1}, a_{i+2}, \ldots a_{i+n}\}) = F_{a\_i}(\text{any n element subset of } \{a_j : j > i\})$ = some constant c. Let $G(a_i) = c$.

Recall that the sequence a is strictly increasing, and thus we can collect it into a set without losing anything: $\{a_i : i \in \omega\}$.

Claim: There is an infinite subset $H` \subseteq \{a_i : i \in \omega\}$ such that G is constant on $H`$.

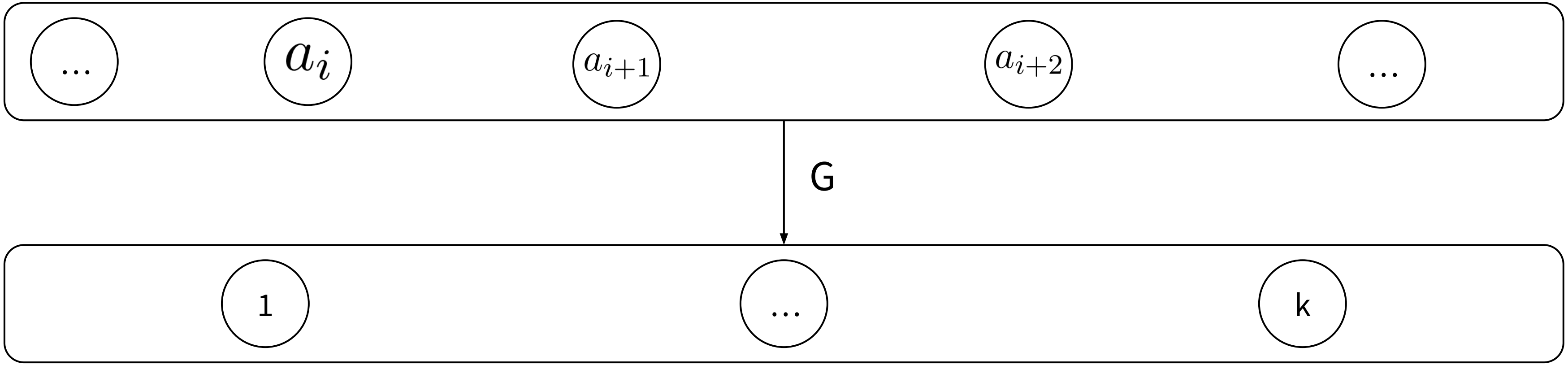
Try to prove this yourself! (don't overthink it)

Proof of Claim

Once again, use the infinite pigeonhole principle! (or the n = 1 case)

We know there as an easy bijection $\{a_i : i \in \omega\}$ to ω, and thus we can apply the infinite pigeonhole principle as G maps H` to the same partition.

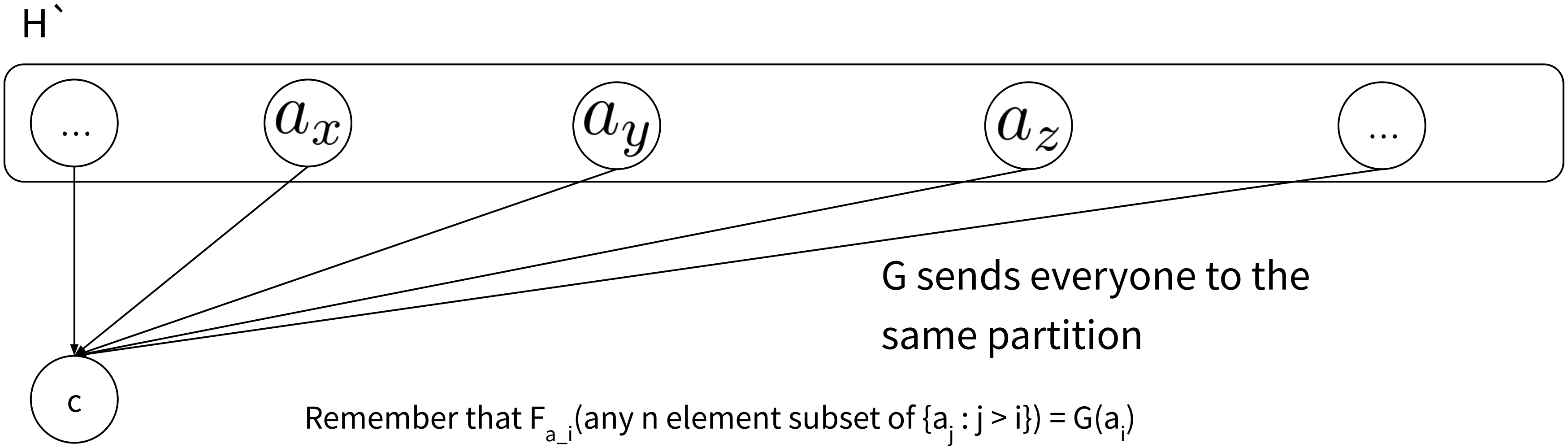
$\{a_i : i \in \omega\}$



Final Claim

We claim that F is constant on $[H`]^{n+1}$.

Consider some $x_1 < x_2 < \ldots < x_{n+1}$ in H`. By the fact that G is constant on H`, we know that $G(x_1) = F_{x\_1}(\{x_2,\ldots,x_{n+1}\})$. And this is true for any choice of $x_1 \ldots x_{n+1}$ in H`.



Remember that F_{a_i}(any n element subset of $\{a_j : j > i\}$) = $G(a_i)$
and since $x_2 \ldots x_{n+1}$ are all $> x_1$, we apply the function and get the result $G(x_1) = F_{x\_1}(\{x_2,\ldots,x_{n+1}\})$.

End of Proof

From that, we can realize two things that are important:

1. Any choice of $x_1 \ldots x_{n+1}$ in H` is sent to the **same** partition
2. $F_{x\_1}(\{x_2,\ldots,x_{n+1}\}) = F(\{x_1,x_2,\ldots,x_{n+1}\})$; (since $F_a(x) = F(\{a\} \cup x)$)

Therefore, F is constant on $[H`]^{n+1}$, which thereby completes the claim and therefore the proof. :)

A Quick Note

This theorem generalizes easily to a variety of interesting results that are foundational to infinitary combinatorics, and sometimes are even the subject of current research.

Obviously, since one proof took so long, we won’t be able to cover it.

Here are some of the aforementioned results:

- Erdős–Rado Theorem (generalizes Ramsey to larger sets)
- Milliken–Taylor Theorem (generalizes to tree structures)
- Large Cardinal Program (active research area)

Thank You

:)