Infinitary Combinatorics

Jason Zhang

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1 Introduction

This serves as an introductory lecture to **infinitary combinatorics**. We will only cover one important result due to time constraints. Additionally, it is almost guaranteed that you will never see this in any math olympiad or competition due to its technicality and scope. Therefore, it is important to treat this lecture as a learning experience or an exercise in proofs, rather than something useful to extract meaningful knowledge for regular olympiad combinatorics.

1.1 Background

We will need to be familiar with set theory and basic combinatorics. We will first discuss the *pigeonhole principle.* Informally: "let n and m be positive integers with $n > m$. If n items are placed into m containers, then at least one container has more than one item." Here is a more formal version:

Proposition 1.0.1 (Pigeonhole Principle)**.** *If n, m are positive integers with $n > m$, then any function $f : \{1, \dots, n\} \to \{1, \dots, m\}$ is not injective.*

Verify for yourself that this indeed is equivalent to the informal version.

Proof. We will prove the informal version since we could easily apply that to obtain the version above. Let $n = m + 1$ and we will place every item into a container. Suppose for contradiction that we placed at most one item in every container. Only m items were placed, and we said there would be $m + 1$, a contradiction. It follows that at least. If $n > m + 1$ the same argument would apply. ☐

We will be covering concepts in Ramsey Theory (an area of combinatorics later). However, it is not important to know about the finite Ramsey Theoretic concepts in this context, although it could bolster your understanding. Regardless, we will not be covering those concepts unless necessary for exploring the infinite case.

2 Infinity

To truly take advantage of infinitary combinatorics, we need to learn about infinity. We will also define $|A|$ to be the number of elements in A. We define the set $\omega = \{0, 1, 2, \dots\}$. You can think of this set as the natural numbers or infinity (as $|\omega|$ is infinite). We will also introduce the notation $[A]^n$ where $n > 0$. This is defined as

$$[A]^n = \{X \subseteq A : |X| = n\}.$$

In other words, the set of all subsets of A with size n (could be seen as "A choose n"). Equivalently, $[A]^n$ is the set of all strictly increasing sequences of numbers $\langle a_1, a_2, \dots, a_n\rangle$ in A.[1]

3 Infinite Principles

This is where the combinatorics actually starts. Let us first consider some important definitions.

Definition 3.1. A partition $X = \{X_i : i \in I\}$ of a set S is a disjoint family of subsets of S such that $\bigcup_{i\in I} X_i = S$.

Definition 3.2. Let X be a partition of $[A]^n$. A set $H \subseteq A$ is *homogeneous* if there exists some i where $[H]^n$ is completely included in X_i. In other words, every n element subset of H is included in the same partition.

[1] Definitions are simplified due to lecture time constraints.

3.1 Ramsey Theory of Infinite Sets

We are inclined to think about the pigeonhole principle on infinite sets. Let us informally state one that makes sense:

Theorem 3.3. *If an infinite set is partitioned into finitely many pieces, then at least one piece is infinite.*

Proof. Let us proceed by contradiction. Assume that the partition of an infinite set S into finite pieces is all finite. However, the union of finite sets is finite. Therefore, $|S| = |\bigcup_i X_i| < |\omega|$, contradiction. □

This is naturally intuitive and keeps the same spirit as the pigeonhole principle. However, we can actually prove a much stronger result.

Theorem 3.4 (Ramsey)**.** *Let n and k be nonzero natural numbers. Every partition $\{X_1, \dots, X_k\}$ of $[\omega]^n$ has an infinite homogeneous set. Equivalently, for every $F : [\omega]^n \to \{1, \dots, k\}$ there exists an infinite $H \subseteq \omega$ such that F is constant on $[H]^n$.*

It turns out that the second equivalent statement is much easier to prove, so we will aim to do that.

Proof. Let us induct over n. For the base case $n = 1$, the theorem is trivial. If you are unsure, consider that $[\omega]^1 = \{\{0\}, \{1\}, \{2\}, \dots\}$. This is actually just the aforementioned infinite pigeonhole principle! We can simply let $H = F^{-1}(i)$ for some $i \leq k$ which would be infinite and homogeneous, and works by the infinite pigeonhole principle.
The inductive case is much harder. Assume it is true for n and we will aim to prove it for $n + 1$. Let F be a function from $[\omega]^{n+1}$ onto $\{1, \dots, k\}$. For every $\alpha \in \mathbb{N}$ (equivalently, $\alpha \in \omega$), let F_α be the function on $[\omega - \{\alpha\}]^n$. Specifically,

$$F_\alpha = F(\{\alpha\} \cup X).$$

From the induction hypothesis we know for all $\alpha \in \omega$ and infinite $S \subseteq \omega$ there exists a $H_\alpha^S \subseteq S - \{\alpha\}$ such that F_α is constant over $[H_\alpha^S]^n$ (there is a bijection $\omega \to S$ and thus with some minor adjustments the n case applies here). Let us construct an infinite sequence $\langle a_i : i \in \omega \rangle$ and let $S_0 = \omega$ and $a_0 = 0$. Now, let us define $S_{i+1} = H_{a_i}^{S_i}$ and a_{i+1} to be the least element of S_{i+1} that is greater than a_i. At this point, we must realize that for all a_i, the function F_{a_i} is constant on $[S_{i+1}]^n$ due to our inductive hypothesis and construction of F_α and S_i. Realize that set $[\{a_j : j > i\}]^n$ is within $[S_{i+1}]^n$ by construction, so F_{a_i} is also constant on it. Let that value of F_{a_i} be $G(a_i)$. There will be some infinite subset $H' \subseteq \{a_i : i \in \omega\}$ where G is constant on H' (as a simple application of the $n = 1$ case or the infinite pigeonhole principle!). We can now say F is constant on $[H']^{n+1}$. This is because for any $x_1 < x_2 < \cdots < x_{n+1}$ in H', the function F_{x_1} is constant on $\{x_2, x_3, \dots, x_{n+1}\}$ (due to the fact that G being constant on H' ensures everything gets sent to the same partition). Recall that $F_{x_1}(\{x_2, x_3, \dots, x_{n+1}\}) = F(\{x_1, x_2, \dots, x_{n+1}\})$, which completes the proof. □

This result can be generalized and expanded, which is the subject of many results in infinitary combinatorics.

4 References

The information from this lecture is adapted from Set Theory The Third Millennium Edition by Thomas Jech.