Infinitary Combinatorics

Jason Zhang

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1 Introduction

This serves as an introductory lecture to **infinitary combinatorics**. It should be noted that many theorems and results will be simplified to a degree. Additionally, it is almost guaranteed that you will never see this in any math olympiad or competition due to its technicality and scope. Therefore, it is important to treat this lecture as a learning experience or an exercise in proofs, rather than something useful to extract meaningful knowledge for regular olympiad combinatorics.

1.1 Background

We will need to be familiar with set theory and basic combinatorics. We will first discuss the *pigeonhole principle*. Informally: "let n and m be positive integers with $n > m$. If n items are placed into m containers, then at least one container has more than one item." Here is a more formal version:

Proposition 1.0.1 (Pigeonhole Principle). *If n, m are positive integers with $n > m$, then any function $f : \{1, \dots, n\} \to \{1, \dots, m\}$ is not injective.*

Verify for yourself that this indeed is equivalent to the informal version. Additionally, try to prove either of the statements (simplest proof is by contradiction). We will also introduce some conventions according to set theoretic definitions. The biggest one is that

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

Notice that $0 \in \mathbb{N}$. This is extremely useful in set theory[1]. For all future mentions of the "natural numbers", we will adhere to that definition. We will be covering concepts in Ramsey Theory (an area of combinatorics later). However, it is not important to know about the finite Ramsey Theoretic concepts in this context, although it could bolster your understanding. Regardless, we will not be covering those concepts unless necessary for exploring the infinite case.

2 Infinity

To truly take advantage of infinitary combinatorics, we need to learn about infinity. The most commonly discussed infinities in set theory are that of *cardinals* and *ordinals*. These often correspond to each other, but they are not the same.

2.1 Ordinals

The idea is to define numbers as sets. The motivation of this idea will become more clear later. For now, we will define any natural number as

$$n = \{0, \dots, n - 1\}.$$

It should be noted that $0 = \varnothing$. Another definition that is slightly more formal is,

$$\alpha = \{\beta : \beta < \alpha\}.$$

From these two equivalent definitions, we can infer that

$$n + 1 = n \cup \{n\}.$$

The last important property of an ordinal is the fact that it is ordered (hence the name). We will define $<$ as so:

$$a < b \text{ if and only if } a \in b.$$

It should be noted that these are all largely informal definitions. However, there is a real formal definition.

[1] And in general is a better way to notate it. If you need just the positive integers then you already have $\mathbb{Z}^+$

Definition 2.1. A set is an *ordinal number* if and only if it is transitive and well ordered by $\in$.

We will elaborate on this definition later. For now, let us consider the first infinite ordinal denoted as ω.[2] It should be noted that $\omega = \mathbb{N}$. Due to the definition of the ordinals, we can show that $\omega < \omega + 1$. The intuition behind this is that ω is representative of the ordering $0 < 1 < 2 < \cdots$ while $\omega + 1$ is representative of $0 < 1 < 2 < \cdots < \omega$. There is no way to "translate" one ordering into another. However, $1 + \omega = \omega$. This is because there is a way to translate the orderings! We can call the element added to the start -1, so we need to find a function from $-1 < 0 < 1 < \cdots$ to $0 < 1 < 2 < \cdots$, which is easy. $f(x) = x + 1$ works as it is bijective and preserves the order. Hence $1 + \omega$ and ω are *order isomorphic*, which means they are equal.

2.1.1 Orderings

We will briefly discuss more general concepts about orderings. Note that ordinals are a very specific case of orderings.

Definition 2.2. A binary relation $\prec$ on a set P is a *partial ordering* if the following conditions hold:

(i) $p \not\prec p$ for all $p \in P$

(ii) If $a \prec b$ and $b \prec c$ then $a \prec c$.

We call $(P, \prec)$ a partially ordered set. A partial order is a *linear order* if $p \preceq q$ or $q \preceq p$ for all $p, q \in P$. You should be familiar with most terminology on ordered sets (maximum, minimum, upper bound, etc.). We will work with two new ones. The *supremum* is the least upper bound. The *infimum* is the greatest lower bound. The supremum of a set X, if it exists, is denoted as $\sup X$. The infimum is $\inf X$.

Definition 2.3. A linear order $\prec$ of a set P is a *well ordering* if every nonempty subset has a least element.

To finish our understanding of an ordinal, we should note that a set S is transitive if and only if for any $x \in S$, $x \subseteq S$.

Theorem 2.4. *If every set can be put into a well order, then there exists a choice function on every collection of nonempty sets.*

Proof. Suppose we are trying to find the choice function of the set S. This is a function $f : S \to \bigcup S$. We know that $\bigcup S$ has a well order. For each $x \in S$, we take $f(x)$ to be the least element in x by the well ordering. □

This means the fact that all sets are well ordered is equivalent to the axiom of choice, a very useful axiom.

2.2 Cardinals

Cardinals measure size. In a very simple example, $|\{0, 1, 2\}| = 3$. We say that

$$|X| = |Y|$$

if and only if there is a bijective function $f : X \to Y$. We call the smallest cardinal infinity $\aleph_0$. Verify that $|\omega + 1| = \aleph_0$.

Let us define the powerset of a set X to be the set of all subsets of X. In other words, $\mathcal{P}(X) = \{A : A \subseteq X\}$.

Lemma 2.5 (Cantor)**.** *For all sets X, $|X| < |\mathcal{P}(X)|$.*

Proof. Define a function $f : X \to \mathcal{P}(X)$. Consider the following set

$$Y = \{x \in X : x \notin f(x)\}.$$

This set is evidently not in the range of f (there is no $z \in X$ where $f(z) = Y$. If there was, then $z \in Y$ if and only if $z \notin Y$, contradiction). Thus f cannot be bijective and $|X| \neq |\mathcal{P}(X)|$. A function $g(x) = \{x\}$ is injective, which implies $|X| \leq |\mathcal{P}(X)|$ (by definition). This completes the proof. □

This particular argument is called the *diagonal argument*, a very powerful proof technique. We can also use this to prove that $|\mathbb{N}| < |\mathbb{R}|$. We will denote $|\mathbb{R}| = \mathfrak{c}$. The infinite cardinal that corresponds to $|\mathbb{N}|$ is $\aleph_0$. The next is $\aleph_1$, then $\aleph_2$, and so on.

Corollary 2.5.1. *For all cardinals κ, $\kappa < 2^\kappa$.*

You can prove that if $|X| = \kappa$, then $|\mathcal{P}(X)| = 2^\kappa$ by considering the set of all functions from $X \to \{0, 1\}$. We will also introduce the notation $[A]^n$ where $n > 0$. This is defined as

$$[A]^n = \{X \subseteq A : |X| = n\}.$$

In other words, the set of all subsets of A with size n. Equivalently, $[A]^n$ is the set of all strictly increasing sequences of ordinals $\langle a_1, a_2, \ldots, a_n \rangle$ in A.

[2] ω is formally defined as the smallest inductive set. An inductive set S satisfies $\varnothing \in S$ and for all $x \in S$, $x \cup \{x\} \in S$.

3 Infinite Principles

This is where the combinatorics actually starts. Let us first consider some important definitions.

Definition 3.1. A partition $X = \{X_i : i \in I\}$ of a set S is a disjoint family of subsets of S such that $\bigcup_{i\in I} X_i = S$.

Definition 3.2. Let X be a partition of $[A]^n$. A set $H \subseteq A$ is *homogeneous* if there exists some i where $[H]^n$ is completely included in X_i. In other words, every n element subset of H is included in the same partition.

3.1 Ramsey Theory of Infinite Sets

We are inclined to think about the pigeonhole principle on infinite sets. Let us informally state one that makes sense: "If an infinite set is partitioned into finitely many pieces, then at least one piece is infinite." This is naturally intuitive and keeps the same spirit as the pigeonhole principle. However, we can actually prove a much stronger result.

Theorem 3.3 (Ramsey)**.** *Let n and k be nonzero natural numbers. Every partition $\{X_1, \dots, X_k\}$ of $[\omega]^n$ has an infinite homogeneous set. Equivalently, for every $F : [\omega]^n \to \{1, \dots, k\}$ there exists an infinite $H \subseteq \omega$ such that F is constant on $[H]^n$.*

It turns out that the second equivalent statement is much easier to prove, so we will aim to do that.

Proof. Let us induct over n. For the base case $n = 1$, the theorem is trivial. If you are unsure, consider that $[\omega]^1 = \{\{0\}, \{1\}, \{2\}, \dots\}$. This is actually just the aforementioned infinite pigeonhole principle! We can simply let $H = F^{-1}(i)$ for some $i \leq k$ which would be infinite and homogeneous, and works by the infinite pigeonhole principle (a proof of which is intuitively obvious).
The inductive case is much harder. Assume it is true for n and we will aim to prove it for $n + 1$. Let F be a function from $[\omega]^{n+1}$ onto $\{1, \dots, k\}$. For every $\alpha \in \mathbb{N}$ (equivalently, $\alpha \in \omega$), let F_α be the function on $[\omega - \{\alpha\}]^n$. Specifically,

$$F_\alpha = F(\{\alpha\} \cup X).$$

From the induction hypothesis we know for all $\alpha \in \omega$ and infinite $S \subseteq \omega$ there exists a $H^S_\alpha \subseteq S - \{\alpha\}$ such that F_α is constant over $[H^S_\alpha]^n$ (there is a bijection $\omega \to S$ and thus with some minor adjustments the n case applies here). Let us construct an infinite sequence $\langle a_i : i < \omega\rangle$ and let $S_0 = \omega$ and $a_0 = 0$. Now, let us define $S_{i+1} = H^{S_i}_{a_i}$ and a_{i+1} to be the least element of S_{i+1} that is greater than a_i. At this point, we must realize that for all a_i, the function F_{a_i} is constant on $[S_{i+1}]^n$ due to our inductive hypothesis and construction of F_α and S_i. Realize that set $\{[a_j : j > i\}]^n$ is within $[S_{i+1}]^n$ by construction, so F_{a_i} is also constant on it. Let that value of F_{a_i} be $G(a_i)$. There will be some infinite subset $H' \subseteq \{a_i : i < \omega\}$ where G is constant on H' (as a simple application of the $n = 1$ case or the infinite pigeonhole principle!). We can now say F is constant on $[H']^{n+1}$. This is because for any $x_1 < x_2 < \cdots < x_{n+1}$, the function F_{x_1} is constant on $\{x_2, x_3, \dots, x_{n+1}\}$ (due to the fact that G being constant on H' ensures everything gets sent to the same partition). Recall that $F_{x_1}(\{x_2, x_3, \dots, x_{n+1}\}) = F(\{x_1, x_2, \dots, x_{n+1}\})$, which completes the proof. □

It should be noted that this theorem implies a similar statement for finite sets, which is the focus of study for Finite Ramsey Theory. We will introduce new notation to provide more general versions of this theorem.

Definition 3.4. We say $\kappa \to (\lambda)^n_k$ means "Every partition of $[\kappa]^n$ into k pieces has a homogeneous set of size λ."

We can then easily restate theorem 3.3 as $\aleph_0 \to (\aleph_0)^n_k$ for positive integers n, k. On another note, we will delete the subscript when $k = 2$. So $\kappa \to (\lambda)^n$ is the same as $\kappa \to (\lambda)^n_2$. There are many generalizations of 3.3 that can be made. We will go over one and sketch a proof idea (mainly because the real proof would be too long and complex to adequately cover in a short lecture like this).

Definition 3.5. For any cardinal κ, let $\beth_0(\kappa) = \kappa$ and $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$.

Definition 3.6. For any cardinal κ, denote κ^+ to be the successor cardinal. This means the smallest cardinal larger than κ.

Theorem 3.7 (Erdős–Rado)**.** $\beth_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa$

Proof. Again, this is a proof sketch and will skip over some steps. Let us think about it for $\kappa = \aleph_0$ as it is the "simplest case" of sorts. The result can easily be generalized to a variety of large cardinals by simplifying replacing specific instances of $\aleph_0$ or similar with κ. Now, let us restate.

$$\beth^+_n \to (\aleph_1)^{n+1}_{\aleph_0}.$$

Note that $\beth_n(\aleph_0) = \beth_n$ and $\aleph_1 = \aleph_0^+$. Let us induct over n again. If $n = 1$, then

$$(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}.$$

Proving this is particularly useful since the inductive step mirrors this case. Suppose $\lambda = (2^{\aleph_0})^+$ and let $F : [\lambda]^2 \to \omega$ be a partition of $[\lambda]^2$ into $\aleph_0$ pieces. We want to find a homogeneous subset $H \subseteq \lambda$ where $|H| = \aleph_1$. We follow a similar proof to theorem 3.3. For every $\alpha \in \lambda$, let F_α be a function over $\lambda - \{\alpha\}$ where $F_\alpha(x) = F(\{\alpha, x\})$. We want to prove a claim stating that there exists a set $A \subseteq \lambda$ where $|A| = 2^{\aleph_0}$ and so that every countable subset (countable means of size $< \aleph_1$) $C \subset A$ and every $u \in \lambda - C$ there is a $v \in A - C$ where F_u and F_v agree on C. The proof of this claim involves constructing an ω_1 sequence of subsets ($A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots$ for $\alpha < \omega_1$). This sequence would "locally" agree to the claim due to the construction of the sequence and the limited number of F_α. Then, we can take $A = \bigcup_{\beta<\omega_1} A_\beta$ to complete the proof of the claim. We then choose from $a \in \lambda - A$ and construct a sequence $\langle x_i : i < \omega_1 \rangle$ in A such that x_0 is arbitrary and given some restriction of $\{x_i : i < \beta\} = C$, x_β is the $v \in A - C$ such that F_v agrees with F_a on C (which exists from the claim). Let $X = \{x_i : i < \omega_1\}$. Consider the function $G : X \to \omega$ where $G(x) = F_a(x)$ (a is the from before). Suppose $\alpha < \beta$. It is true that $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha) = F_a(x_\alpha) = G(x_\alpha)$ (remember that F_{x_β} would agree with F_a on all x_i for $i < \beta$). The range of$G$ is countable, which means there must exists some $H \subseteq X$ of size $\aleph_1$ where G is constant on H (another pigeonhole principle). From here it would follow that F is constant on $[H]^2$ (think about how G translates to F). For the inductive case, it is largely the same thing except we generalize from just "2" to "n". For example, instead of C just being countable ($\leq \aleph_0 = \beth_{1-1}$), $|C|$ would be $\leq \beth_{n-1}$. Also, $F_\alpha(x) = F(x \cup \{\alpha\})$, and many instances of C is replaced with $[C]^n$ among other changes. Finally, instead of using an uncountable pigeonhole principle to show that G is constant on H, we use the induction hypothesis. □

3.2 Deep Results

This section consists of a rough survey of advanced results and current research.

3.2.1 Trees

Many topics in infinitary combinatorics can be interpreted by trees. Let us first understand the set theoretic definition of a tree.

Definition 3.8. A tree is a partially ordered set $(T, \prec)$ such that for every $x \in T$, the set $\{y : y \prec x\}$ is well ordered by $\prec$.

In other words, a tree is an ordered set that is connected and every branch is well ordered. Note the following for a tree T:

$$\begin{aligned} o(x) &= \text{the order type of } \{y : y < x\} \\ \alpha\text{th level} &= \{x : o(x) = \alpha\} \\ \text{height}(x) &= \sup\{o(x) + 1 : x \in T\} \end{aligned}$$

A branch in T is a maximally linear ordered subset of T. The *length* of a branch b is order-type of b. An a-branch of length a. An antichain $A \subset T$ is a set where any two distinct elements are not comparable (neither $x \prec y$ or $y \prec x$).

Definition 3.9. A tree T is a *Suslin Tree* if

(i) The height of the tree is ω_1

(ii) Every branch is at most countable

(iii) Every antichain is at most countable

There is a good reason to learn about *Suslin Trees* which we will cover later.

Definition 3.10. An *Aronzajn tree* is a tree of height ω_1 all of whose levels are at most countable and which has no uncountable branches.

Theorem 3.11 (Aronzajn)**.** *There exists an Aronzajn tree.*

Again, we will only sketch the proof idea.

Proof. We want to construct a tree T whose elements are bounded increasing infinite sequences of rational numbers. For $x, y \in T$, we define $x \leq y$ to be $x \subseteq y$ (in other words, y is an extension of x). If $y \in T$ and x is some segment that includes the start of y, then $x \in T$. Yet, there are no ω_1 branches since that would mean an ω_1 increasing sequence of rationals, which isn't possible since there are only $\aleph_0$ rationals. T would be an Aronzajn tree. The actual construction of T is beyond this lecture. □

3.2.2 Combinatorial Principles in L

Let us first understand what V is before we understand L. V is the *proper class* of all sets. We distinguish it to be a *proper class* instead of a set because a set of all sets is inconsistent in most set theories. More formally, V is defined as follows:

Definition 3.12. Let V_α be a rank of V where α is an ordinal. We define

(i) $V_0 = \varnothing$

(ii) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$

(iii) $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$ where λ is a limit ordinal[3].

And finally $V = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$. In other words, V is in the union of all ranks.

It can be shown that our two definitions are equivalent. L (named the constructible universe) is defined similarly to V with a very slight difference.

Definition 3.13. Define $\mathcal{P}_{\mathrm{Def}}(X)$ to be set of all subsets of X definable from first order formulas with parameters from X.

Think of this as using only subsets that you can properly express. L is simply the same as V but with $\mathcal{P}_{\mathrm{Def}}$ instead of $\mathcal{P}$. This "tames" the universe V, which could have wild sets. L happens to be a really nice model for ZFC (the version of set theory we have been working in), meaning that L satisfies all the axioms of ZFC. But also, L can have specific interesting combinatorial behaviors. Without diving too deep, here is a general list of some properties in L:

(i) For all infinite cardinals κ, $2^\kappa = \kappa^+$ (known as the generalized continuum hypothesis).

(ii) There exist *special* Aronzajn trees, which can be decomposed into countable antichains.

(iii) There exist "Kurepa trees", trees of height ω_1, where levels are countable and there are $> \omega_1$ many branches.

(iv) You can define a well ordering for $\mathbb{R}$.

(v) Much more that are hard to explain without diving into too much background (one for example is where you can "predict" subsets of ω_1. This is called $\Diamond_\kappa$).

3.2.3 Metamathematics

There are some interesting metamathematical considerations to be held. One such is that the Suslin Tree (c.f. 3.9) is independent of ZFC. That is, some models have a Suslin Tree existing and others do not. L has a lot of "weird" things going on as well, as it is a model and so specific principles are true in it that aren't true in other models. These principles are at the heart of current research (e.g., large cardinal program). However, the subject of model theory is often nuanced and so we will not cover much. Just know that metamathematics and essentially philosophy start to play a role.

4 Exercises

Problem 1. Prove that every infinite linearly ordered set either has an infinite increasing sequence of elements or has an infinite decreasing sequence of elements

Problem 2. Complete the proof of theorem 3.7 both by finishing the inductive case and restructuring to accommodate general large cardinals.

5 References

The information from this lecture is adapted from Set Theory The Third Millennium Edition by Thomas Jech.

[3] You can think of this is a "multiple of ω"