Free Monoids

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1 Monoids

Definition 1.1. We say that a set S equipped with a binary operation $\bullet : S \times S \to S$ is a *monoid* iff $\bullet$ is associative over all elements of S and there exists some $e \in S$ such that $e \bullet a = a \bullet e = a$ for all $a \in S$.

Equivalently, a monoid is simply a *semigroup* with the identity condition added. And because a semigroup is just a *magma* with the property of associativity, we can conclude that a monoid is closed under $\bullet$. We can also think of e as a unique 0-ary operator (in other words, constant) which helps us identify the monoid. So we would write $(S, \bullet, e)$ to specify a specific monoid. Below is a list of monoids:

(i) $(\mathbb{N}, +, 0)$

(ii) $(\mathbb{Z}^+, \times, 1)$ where $\times$ is the usual multiplication over the integers.

(iii) For all sets S, $(\mathcal{P}(S), \cup, \varnothing)$ where $\mathcal{P}(S)$ is the set of all subsets of S.

(iv) All singleton sets closed under a binary operation.

It is quite easy to find more, a lot of algebraic structures satisfy it (any group works). The Category Theorist can give us a more general perspective; however, it is far too powerful to be of useful consideration here.

2 Alphabets and Kleene Closure

Definition 2.1. Let Σ denote an alphabet of characters and ε denote the empty string.

For example, we could have $\Sigma = \{a, b\}$. For our purposes, a string can be considered as a finite ordered tuple of symbols and concatenation simply merges the tuples. We will come back with a better definition later.

Definition 2.2. The Kleene Star applied to a set of symbols S is generated by

(i) $S_0 = \{\varepsilon\}$

(ii) $S_{i+1} = \{ab : a \in S_i, b \in S\}$ via concatenation.

(iii) $S^* = \bigcup_{i \geq 0} S_i$

And S^* is the Kleene Closure (or Star) on S.

3 Free Monoid

We can make an equivalent definition of strings and $*$ that is more elegant.

Definition 3.1. For a set A, the free monoid A^* is a monoid operating over all finite sequences of A with concatenation as the binary operator and the empty sequence ε as the identity element.

Therefore, *a string is simply a element of the free monoid* Σ^*. And indeed, a string being a finite sequence of elements of Σ is no different than being an ordered tuple of symbols in Σ. So the Kleene Star on strings in the usual sense is really just a specific version of the free monoid on an alphabet set Σ. This does imply we see free monoids in other places, and indeed we do. If we can find an isomorphism from a monoid to a free monoid, we can conclude the first one is also free.

Theorem 3.2. $(\mathbb{N}, +, 0)$ *is a free monoid.*

Proof. Let $\Sigma = \{a\}$ or a singleton set. Consider the free monoid acting on Σ^* and denote it M. We can find an isomorphism $g : (\mathbb{N}, +, 0) \to M$ where $g(n) = a^n$. This is bijective and preserves the other conditions: $g(0) = \varepsilon$ and $g(n + m) = g(n)g(m)$. □