Partial Derivative Chain Rule for 3D

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§1 Background

Let f be a function of the form $f(x, y) =$ some equation. We define the following as so:

Definition 1.1. We notate $(\frac{\partial f}{\partial x})_y$ to be the derivative of f with respect to x and with y held constant.

We will be treating *differentials* (i.e., $\mathrm{d}x$) as a variable just like we might with x. It should be noted that $\mathrm{d}x$ is the *change* in x.

Let $\vec{r} = \langle x, y\rangle$ and $\nabla f = \langle \left(\frac{\partial f}{\partial x}\right)_y, \left(\frac{\partial f}{\partial y}\right)_x \rangle$. Then

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{r}$$

This equation allows us to recover much useful information, such as the tangent plane. More importantly, it acts like a "total derivative". The relation between $\mathrm{d}f$ and $\mathrm{d}x$, $\mathrm{d}y$ is explicit.[1] We can see the utility for this in an example.

Consider the surface $x^2y + 3y^2z + 7z^2x = 5$ near $(1, 1, -1)$. Find the following:

(i) $\left(\frac{\partial z}{\partial x}\right)_y$

(ii) $\left(\frac{\partial x}{\partial y}\right)_z$

(iii) $\left(\frac{\partial z}{\partial x}\right)_{x^2y}$

(iv) $\left(\frac{\partial y}{\partial x}\right)_{x^3y^2z}$

Solution. Let us first do some analysis on this surface. Instead of trying to recover this into a function of x, y to z, we will instead consider it as a function of x, y, z that sends to some f[2]. So $f(x, y, z) = x^2y + 3y^2z + 7z^2x$, and we are just considering the *level surface* for which $f = 5$. Now, apply $\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{r}$. We get

$$\mathrm{d}f = (2xy + 7z^2)\mathrm{d}x + (x^2 + 6yz)\mathrm{d}y + (3y^2 + 14zx)\mathrm{d}z.$$

We want $f = 5$, so the change in f: $\mathrm{d}f = 0$. Therefore, we obtain

$$0 = (2xy + 7z^2)\mathrm{d}x + (x^2 + 6yz)\mathrm{d}y + (3y^2 + 14zx)\mathrm{d}z.$$

[1]And this applies to the single variable calculus as well.
[2]In fact, we must do this since this is not a function in 3D

Let us now plug in the appropriate values for x, y, z. We obtain

$$0 = 9\mathrm{d}x - 5\mathrm{d}y - 11\mathrm{d}z.$$

(i) We are holding y constant. So $\mathrm{d}y = 0$. Therefore, $0 = 9\partial x - 5(0) - 11\partial z$.[3] Subtracting and dividing accordingly gives us $\left(\frac{\partial z}{\partial x}\right)_y = \frac{9}{11}$.

(ii) Now, we hold z to be constant so $\mathrm{d}z = 0$. $\left(\frac{\partial x}{\partial y}\right)_z = \frac{5}{9}$.

(iii) We want x^2y to be constant. Therefore,

$$\begin{aligned} \mathrm{d}(x^2y) = 2xy\mathrm{d}x + x^2\mathrm{d}y = 0 \\ 2\partial x + \partial y = 0 \end{aligned}$$

Now, we want to eliminate any ∂ys from the equation since our final result is a relation between ∂z and ∂x. Notice that $\partial y = -2\partial x$ by our above equation. Let us plug into any ∂ys. We receive

$$\begin{aligned} 0 &= 9\partial x - 5(-2\partial x) - 11\partial z \\ 0 &= 19\partial x - 11\partial z \\ 11\partial z &= 19\partial x \\ \left(\frac{\partial z}{\partial x}\right)_{x^2y} &= \frac{19}{11}. \end{aligned}$$

(iv) Left as an exercise. $\left(\frac{\partial y}{\partial x}\right)_{x^3y^2z} = -\frac{8}{9}$.

§2 Chain Rule

We have already brushed upon the idea behind the chain rule, but we will formalize it now. Note that example we are about to run through can be generalized.

Let $f(x, y) = x^3y^2 + 2xy^3$ with $x(s, t) = s^2t - 3t$ and $y(s, t) = s^2 + 6t$. Find the following given $s = 2$, $t = -1$:

(i) $\left(\frac{\partial f}{\partial s}\right)_t$

(ii) $\left(\frac{\partial f}{\partial s}\right)_x$

(iii) $\left(\frac{\partial f}{\partial x}\right)_t$

Solution. Let us first cover the textbook definition of the chain rule.

Textbook Definition.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

[3] Notice we are writing with partials (∂) now. The exact moment when the partials should come in is a bit ambiguous, but in general it should be when we decide to hold something constant.

This definition is not elegant and messy. Furthermore, it does not drive any sort of intuition. We will look for better. Let us derive three equations, all centered around the $\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{r}$ idea.

$$\begin{aligned}
\mathrm{d}f &= (3x^2y^2 + 2y^3)\mathrm{d}x + (2x^3y + 6xy^2)\mathrm{d}y \\
\mathrm{d}x &= 3st\mathrm{d}s + (s^2 - 3)\mathrm{d}t \\
\mathrm{d}y &= 2s\mathrm{d}s + 6\mathrm{d}t
\end{aligned}$$

Plugging everything in accordingly gives us our textbook definition. However, we can do better. First, let us establish that

$$\begin{aligned}
x(2, -1) &= -1 \\
y(2, -1) &= -2 \\
\mathrm{d}f &= -4\mathrm{d}x - 20\mathrm{d}y \\
\mathrm{d}x &= -4\mathrm{d}s + \mathrm{d}t \\
\mathrm{d}y &= 4\mathrm{d}s + 6\mathrm{d}t
\end{aligned}$$

Now we are well equipped to start the problems.

(i) Recall that we want $\left(\frac{\partial f}{\partial s}\right)_t$. Plug into $\mathrm{d}x$ and $\mathrm{d}y$ accordingly.

$$\begin{aligned}
\mathrm{d}f &= -4(-4\mathrm{d}s + \mathrm{d}t) - 20(4\mathrm{d}s + 6\mathrm{d}t) \\
\mathrm{d}f &= -64\mathrm{d}s - 124\mathrm{d}t.
\end{aligned}$$

Hold t constant, $\mathrm{d}t = 0$. Solve to obtain $\left(\frac{\partial f}{\partial s}\right)_t = -64$.

(ii) Hold x constant, so $\mathrm{d}x = 0 = -4\partial s + \partial t$. We want a final equation with just f and s, no y or t. We can eliminate t first with $\partial t = 4\partial s$. Then, we substitute $4\partial s$ into ∂t in all places $\partial y = 28\partial s$. Now, finish the equation to give us $\partial f = 0 - 20(28\partial s) \to \left(\frac{\partial f}{\partial s}\right)_x = -560$.

(iii) Left as an exercise. This process is very similar to the above, just with different things held constant and eliminated. $\left(\frac{\partial f}{\partial x}\right)_t = 16$.