Multivariable Calculus Chapter 16 Lecture Notes

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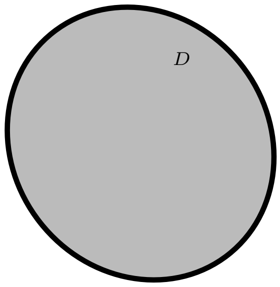
Contents

Disclaimer

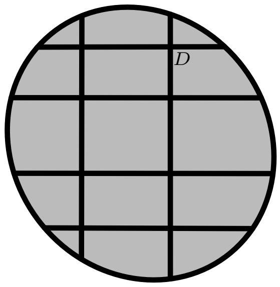
This document was last updated on November 14th, 2025. DM Jason to see if you have the latest version. Many of the commands and styles here were made by Senan Sekhon.

1 Double Integrals

When studying double integrals and beyond, it is important to think of them as some sort of Riemann Sum. Consider some domain D



We would like to partition the domain into smaller shapes like as shown:



We may choose to partition however we like as long as we are partitioning into *areas*. Eventually, the pieces will be infinitely small. At this point, multiplying each piece (area) by its height $f(x, y)$ (length), will give us a volume. We write

$$\iint_D f(x,y) \, \mathrm{d}A$$

for this volume. The differential $\mathrm{d}A$ is our little area pieces. More formally,

Definition 1.1.

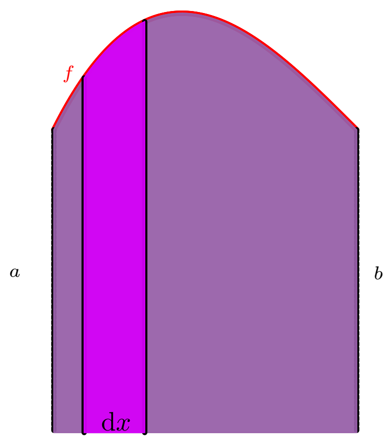
$$\iint_D f(x,y) \, \mathrm{d}A = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta A.$$

1.1 Differentials

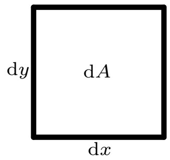
Before we can calculate double integrals effectively, we must first understand the differential in depth. Without a proper understanding of the differential element, we are at a massive disadvantage. Consider the equation

$$\int_a^b f(x) \, \mathrm{d}x.$$

This calculates an area. We are given an infinitesimal $\mathrm{d}x$ which represents a *length* unit. $f(x)$ is also a *length*.



In 3D, our differentials are **areas**.



This shows that dA = dxdy. Hence,

$$\iint f(x,y)\ \mathrm{d}A = \iint f(x,y)\ \mathrm{d}x\mathrm{d}y = \iint f(x,y)\ \mathrm{d}y\mathrm{d}x.$$

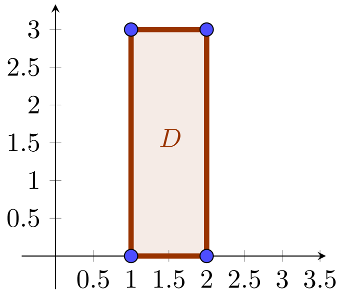
This is not the only way to recover dA, however. In fact, any set of differentials that forms a differential area works.

1.2 Simple Integration

Example. If D is the rectangle with vertices $(1,0),(1,3),(2,0),(2,3)$, evaluate

$$\iint_D x^2y\ \mathrm{d}A.$$

Solution. The first thing you should always do is sketch a diagram of the region domain.



Now, let us consider a thought experiment. Suppose we have a magic function $g(x)$ that tells us the area from $y=0$ to $y=3$ for a fixed value of x. What would the answer to our question be? Indeed, it would simply be

$$\int_1^2 g(x)\ \mathrm{d}x.$$

Can we find such a function? Of course we can, it is

$$\int_0^3 x^2y\ \mathrm{d}y.$$

Here, we treat x as some sort of fixed constant value, no different than 5. Thus, the answer to our problem is evaluating

$$\int_1^2 \left(\int_0^3 x^2y\ \mathrm{d}y\right)\ \mathrm{d}x.$$

Notice that we have gotten the $\iint f(x,y)\ \mathrm{d}y\mathrm{d}x$ form, verifying that our computation will be mathematically sound. From here, we can simply evaluate the inner and outer integral.

$$\begin{aligned}\int_1^2 \left(\int_0^3 x^2y\ \mathrm{d}y\right)\ \mathrm{d}x &= \int_1^2 \left(\frac{x^2y^2}{2}\right)\Bigg|_{y=0}^{y=3}\ \mathrm{d}x\\ &= \int_1^2 \frac{9x^2}{2}\ \mathrm{d}x\\ &= \left(\frac{3x^3}{2}\right)\Bigg|_{x=1}^{x=2}\\ &= 12-\frac{3}{2}\\ &= \boxed{\frac{21}{2}}\end{aligned}$$

□

Going back to a previous point about evaluating dxdy vs. dydx, we can find an important result.

Theorem 1.2 (Fubini's Theorem)**.** Let f be integrable over a rectangular region $D=\{(x,y) : x\in[a,b]; y\in[c,d]\}$.

Then,

$$\iint_D f(x,y)\ \mathrm{d}A = \int_a^b \int_c^d f(x,y)\ \mathrm{d}y\mathrm{d}x = \int_c^d \int_a^b f(x,y)\ \mathrm{d}x\mathrm{d}y.$$

This theorem can be generalized quite a bit, but for the purposes of this lecture the only thing you need to know is that you shouldn't be afraid to switch the integrals as long as you can do it properly. To illustrate, we will solve the above example again the other way around.

Solution.

$$\begin{aligned}\int_0^3 \left(\int_1^2 x^2y\ \mathrm{d}x\right)\ \mathrm{d}y &= \int_0^3 \left(\frac{x^3y}{3}\right)\Bigg|_{x=1}^{x=2} \mathrm{d}y \\ &= \int_0^3 \frac{7y}{3}\ \mathrm{d}y \\ &= \left(\frac{7y^2}{6}\right)\Bigg|_{y=0}^{y=3} \\ &= \boxed{\frac{21}{2}}\end{aligned}$$

□

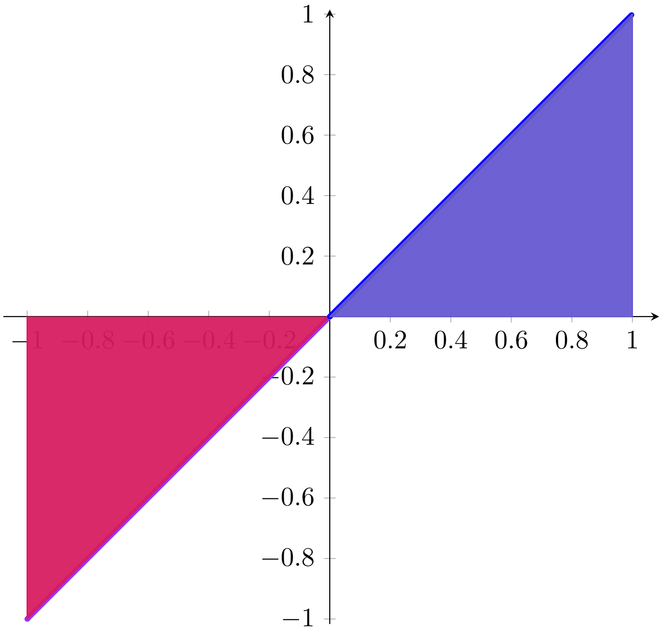
1.3 Symmetry

We will illustrate with an example from AP Calculus.

Example. Evaluate

$$\int_{-1}^1 x\ \mathrm{d}x.$$

Solution. We could expand it out and do a bunch of integration nonsense, but we have a much nicer solution. Let us sketch the graph.



Notice that the pink and blue areas are the same, except for the fact that the pink area is negative. Since the integral is signed area, we know the two simply cancel out. Therefore, the answer is $\boxed{0}$. □

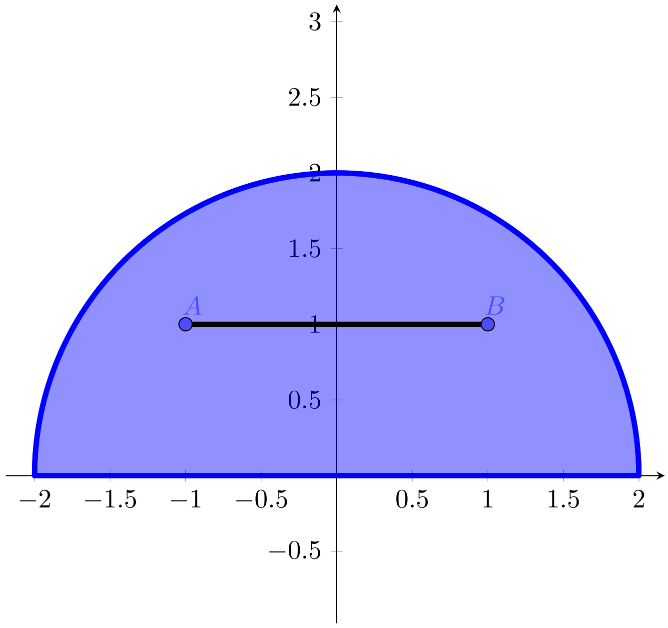
⚠ This does not work on every function! It mostly applies to odd functions when the integral bounds are $-a$ to a. **ALWAYS** sketch a diagram to make sure stuff cancels!

For even functions, the area would double. So $\int_{-1}^1 x^2\ \mathrm{d}x = 2\int_0^1 x^2\ \mathrm{d}x$. Now, let us approach a 3D function.

Example. Consider the region D bounded by $y = \sqrt{4-x^2}$ and the x-axis. Find

$$\iint_D x^3y + xy^2 + y\ \mathrm{d}A.$$

Solution. As always, sketch the region.



Notice the points A and B exhibit symmetry. That is, for odd functions **in x**, $-f(A) = f(B)$. So integrating would cancel them out. Therefore,

$$\iint_D \cancelto{0}{x^3 y} + \cancelto{0}{xy^2} + y \text{ d}A.$$

Notice that the y does not cancel out since there is no y-symmetry. We will now consider the integral

$$\int_{-2}^{2} \left(\int_0^{\sqrt{4-x^2}} y \text{ d}y \right) \text{d}x = \int_{-2}^{2} \text{d}x \int_0^{\sqrt{4-x^2}} \text{d}y \ (y).$$

Notice that we have a function of x in one of the integral bounds. This is perfectly fine, we still treat x as a constant. The only important thing to note is that if we decide to swap the integrals, we will have to change things up since our inner integral would depend on y instead. More specifically, we would write

$$\int_0^2 \text{d}y \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \text{d}y \ (y).$$

You can look at the region to verify that these are equal. We will evaluate the first integral because it is easier.

$$\begin{aligned} \int_{-2}^{2} \text{d}x \int_0^{\sqrt{4-x^2}} \text{d}y \ (y) &= \int_{-2}^{2} \text{d}x \left(\frac{y^2}{2} \right) \Bigg|_{y=0}^{y=\sqrt{4-x^2}} \\ &= \int_{-2}^{2} \frac{4-x^2}{2} \text{ d}x \\ &= \boxed{\frac{16}{3}} \end{aligned}$$

□

Sometimes, swapping makes things much easier, especially when one way doesn't have an elementary antiderivative.

2 Average Values

Example. Find the average value of $[1, 2, 3, 4, 5]$.

Solution. We can think of the answer as $\frac{1+2+3+4+5}{1+1+1+1+1} = 3$. □

This way of thinking sees its advantages later. Specifically, the form $\frac{\text{adding nums}}{\text{adding 1s}}$. This can be generalized.

Lemma 2.1. The average value of an integrable function $f(x)$ over the interval $[a, b]$ is

$$\frac{\int_a^b f(x) \text{ d}x}{\int_a^b 1 \text{ d}x}.$$

You may remember from AP Calculus that the average value is $\frac{1}{b-a} \int_a^b f(x) \text{ d}x$, which is true. $\int_a^b 1 \text{ d}x = b - a$. But the idea is that we can take this intuition into 3D functions:

Lemma 2.2. The average value of an integrable function $f(x,y)$ over the domain D is

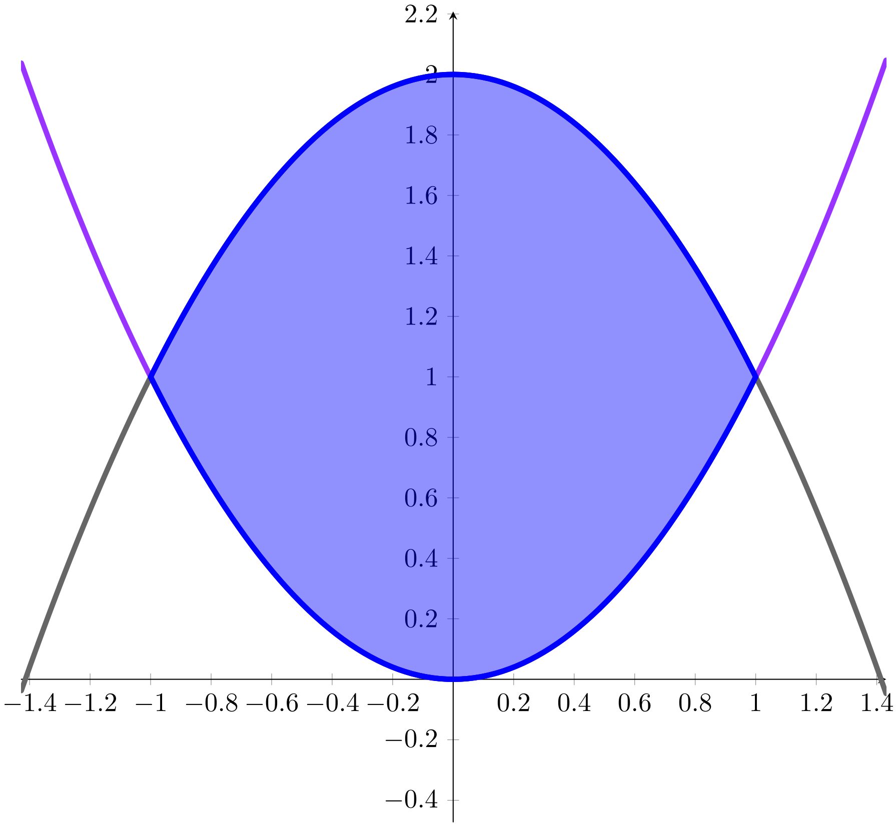
$$\frac{\iint_D f(x,y)\ \mathrm{d}A}{\iint_D 1\ \mathrm{d}A}.$$

This is known as an "area-weighted average", which is what we mean by default. If $A(D)$ is the area of the domain, then this average equates to

$$\sum_{i,j} f(x_i, y_j)\frac{\Delta A_{i,j}}{A(D)}.$$

Example. Find the average value of x^2y on the region bounded by $y = x^2$ and $y = 2 - x^2$.

Solution. Consider the domain region D.



The curves intersect at $x = \pm 1$. There is also symmetry in x. While nothing cancels, an even function like x^2y or 1 would double. Now,

$$\langle x^2y \rangle = \frac{\iint_D x^2y\ \mathrm{d}A}{\iint_D 1\ \mathrm{d}A}.$$

We will compute the two integrals separately.

$$\begin{aligned}\iint_D 1\ \mathrm{d}A &= \int_{-1}^{1} \mathrm{d}x \int_{x^2}^{2-x^2} \mathrm{d}y\ (1)\\ &= 2\int_0^1 (2-2x^2)\mathrm{d}x\\ &= \frac{8}{3}.\end{aligned}$$

And the second integral,

$$\begin{aligned}\iint_D (x^2y)\ \mathrm{d}A &= \int_{-1}^{1} \mathrm{d}x \int_{x^2}^{2-x^2} \mathrm{d}y\ (x^2y)\\ &= 2\int_0^1 \left(\frac{x^2y^2}{2}\right)\Bigg|_{y=x^2}^{y=2-x^2} \mathrm{d}x\\ &= \int_0^1 (x^2(2-x^2)^2 - x^2x^4)\mathrm{d}x\\ &= \frac{8}{15}.\end{aligned}$$

We can then conclude the answer is $\boxed{\frac{1}{5}}$. □

2.1 Centroid and Center of Mass

Definition 2.3. The *centroid*, or geometric center, of a region is the area-weighted average of a region of position in the region.

$$\langle \vec{R} \rangle = \sum_{i,j} \vec{r}_{i,j} \frac{\Delta A_{i,j}}{A}.$$

Equivalently,

$$\langle x \rangle = \frac{\iint_D x \ \mathrm{d}A}{\iint_D 1 \ \mathrm{d}A}; \langle y \rangle = \frac{\iint_D y \ \mathrm{d}A}{\iint_D 1 \ \mathrm{d}A}.$$

When approaching these problems, you should calculate the three integrals separately and combine for your answer. Additionally, you should verify that your centroid makes sense by seeing if it roughly falls in the middle of your region.

Exercise. Find the coordinates of the centroid of the region bounded by $y = x^2$ and $y^3 = x$.

Solution. Left as an exercise. $\boxed{\left(\frac{3}{7}, \frac{12}{25}\right)}$ □

2.2 Density

We will now consider the effects of introducing a density factor. Once again, consider the humble average value example. Let's say we want to find the average between 1 and 2, but we give 1 a weighting of 100 and 2 a weighting of just 1. This is like having 100 different 1 elements to average. The average is exactly $\frac{1 \cdot 100 + 2 \cdot 1}{100+1}$. So if σ tells us the weighting, we would apply σ to every element on the top and bottom. That is,

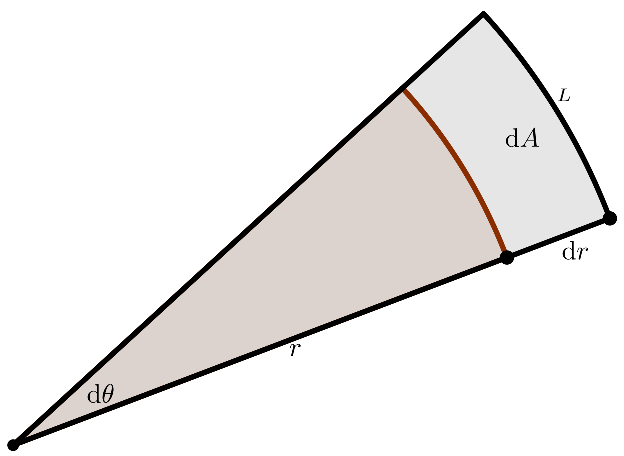
$$\langle x \rangle = \frac{\iint_D x\sigma \ \mathrm{d}A}{\iint_D \sigma \ \mathrm{d}A}; \langle y \rangle = \frac{\iint_D y\sigma \ \mathrm{d}A}{\iint_D \sigma \ \mathrm{d}A}.$$

3 Polar Differentials

Sometimes, it is way too hard to use the cartesian $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$, especially when we deal with circles. Circles can be divided nicely with circular arcs, not rectangles. So we might want to use r and θ instead.

Lemma 3.1. $\mathrm{d}A = r \ \mathrm{d}r\mathrm{d}\theta$.

Proof. Consider the following diagram:



When $\mathrm{d}r$ and $\mathrm{d}\theta$ are infinitesimally small, the region $\mathrm{d}A$ is approximately rectangular, so the area $\mathrm{d}A = L \ \mathrm{d}r$. Now, we know the circumference of the circle is $2\pi r$. The length of this specific arc is $L = \cancel{2\pi} r \frac{\mathrm{d}\theta}{\cancel{2\pi}}$. The "$r$" in question here is just the regular r since the differential $\mathrm{d}r$ is infinitely small. Therefore, $\mathrm{d}A = r \ \mathrm{d}r\mathrm{d}\theta$. □

Remark. This makes a lot of sense when you think about it. Both r and $\mathrm{d}r$ are length units, while $\mathrm{d}\theta$ is not. Therefore, to form an area unit, we need both r and $\mathrm{d}r$. Just using $\mathrm{d}r\mathrm{d}\theta$ doesn't make any sense.

Let us do an example to cover all the knowledge we have gathered thus far.

Example. Find the center of mass of the region bounded by $y = 0$ and $y = \sqrt{4 - x^2}$ given the density is $\sigma = 2 - x$.

Solution. The diagram is just the same semicircle from an example above. Draw it out yourself if you need it. Realize that there is symmetry in x. Our first integral will be associated with the x-numerator. This is

$$\iint_D x\sigma \ \mathrm{d}A = \iint_D x(\cancelto{0}{2} - x) \ \mathrm{d}A = -\iint x^2 \ \mathrm{d}A.$$

Let us first try to naively use Cartesian coordinates. We will eventually receive $-\int_{-2}^{2} \mathrm{d}x \ (x^2\sqrt{4-x^2})$ which is ugly. Let us instead approach this from the polar perspective:

$$-\iint_D x^2 = -\int_0^2 r\mathrm{d}r \int_0^\pi \mathrm{d}\theta \ (r^2 \cos^2\theta).$$

Notice that in the inner integral, r^2 is constant so we can factor it out. We will be left with

$$-\int_0^2 r\mathrm{d}r \ (r^2) \int_0^\pi \mathrm{d}\theta \ (\cos^2\theta),$$

at which point we can realize that $\int_0^\pi \mathrm{d}\theta \ (\cos^2\theta)$ term is constant in the outer integral, so we can factor it out. This leaves us with

$$-\left(\int_0^2 r^3 \ \mathrm{d}r\right)\left(\int_0^\pi \cos^2\theta \ \mathrm{d}\theta\right).$$

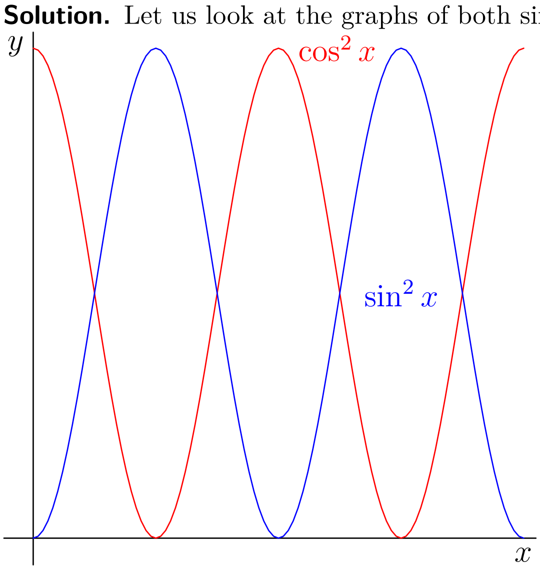
Upon evaluating each integral separately, we get -2π. The remaining two integrals are left as an exercise. The answer to this problem is $\boxed{\left(-\frac{1}{2}, \frac{8}{3\pi}\right)}$. □

3.1 Trig Integrals

Example. Evaluate the integral

$$\int_0^{2\pi} \cos^2 x \ \mathrm{d}x.$$

Solution. Let us look at the graphs of both $\sin^2 x$ and $\cos^2 x$.



The average of $\sin^2 x$ and $\cos^2 x$ is $\frac{\sin^2 x + \cos^2 x}{2} = \frac{1}{2}$. Notice that on whole quarter intervals, the functions $\sin^2 x$ and $\cos^2 x$ behave like their averages. Thus, the answer is the length of the interval multiplied by $\frac{1}{2}$. The answer is therefore $\boxed{\pi}$. □

Theorem 3.2. More generally, we can say that if T represents a whole number of quarter periods of $\sin(cx)$ for constant c, then

$$\int_0^T \sin^2(cx) \ \mathrm{d}x = \int_0^T \cos^2(cx) \ \mathrm{d}x = \frac{1}{2}T.$$

The proof follows a similar idea to the above example. We can use this result to derive two more integrals that can be used to do problems.

Lemma 3.3.
$$\int_0^\pi \sin^2(x)\cos^2(x)\ \mathrm{d}x = \frac{\pi}{8}.$$

Proof. Recall that $\sin 2x = 2\sin x \cos x$ and thus $\sin^2 2x = 4\sin^2 x \cos^2 x$. Now, behold! Simple algebraic manipulation.

$$\begin{aligned}\int_0^\pi \sin^2(x)\cos^2(x)\ \mathrm{d}x &= \int_0^\pi \sin^2(x)\cos^2(x)\ \mathrm{d}x \\ &= \frac{1}{4}\int_0^\pi \sin^2(2x)\ \mathrm{d}x \\ &= \frac{\pi}{8}.\end{aligned}$$ □

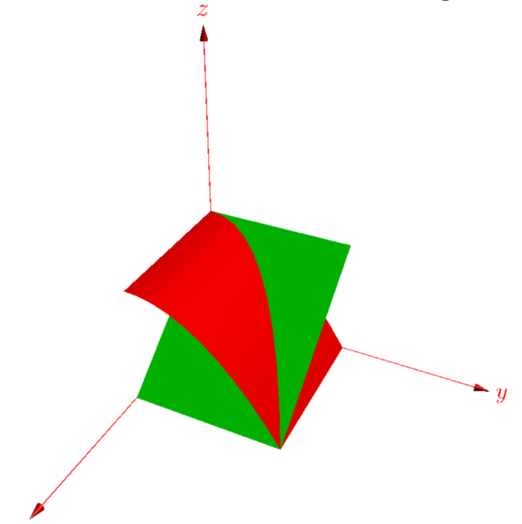
The other integral, $\int_0^\pi \cos^4 x\ \mathrm{d}x$, equals $\int_0^\pi \cos^2 x(1-\sin^2 x)\ \mathrm{d}x$ which can be split and solved using the above identities.

4 Triple Integrals

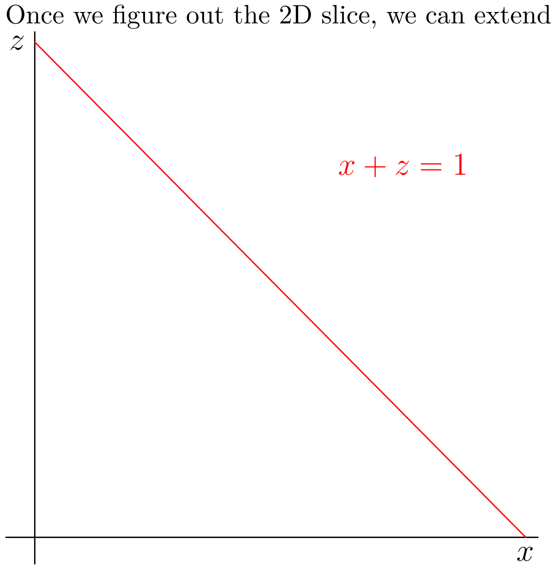
Triple integrals are hard to visualize as we are mortal beings. You may receive graphs on assessments, or you may not. It is important to draw diagrams and to consider 2D cases. Evaluating triple integrals is usually no different than double integrals, just with extra steps. However, setting them up can be challenging.

Example. Clarify an iterated integral covering volume V bounded by $x = 0$, $z = 0$, $x + z = 1$, and $z = 1 - y^2$
$$\iiint_V \mathrm{d}V.$$

Solution. Let us consider the entire region first.



We specifically care about the region under the green and red surfaces in the first octant. Our approach to solving these problems involves first selecting a "base". This is a 2D slice of the region that is hopefully nice to work with. Once we figure out the 2D slice, we can extend it to the 3D. In our case, let us consider the xz-plane.



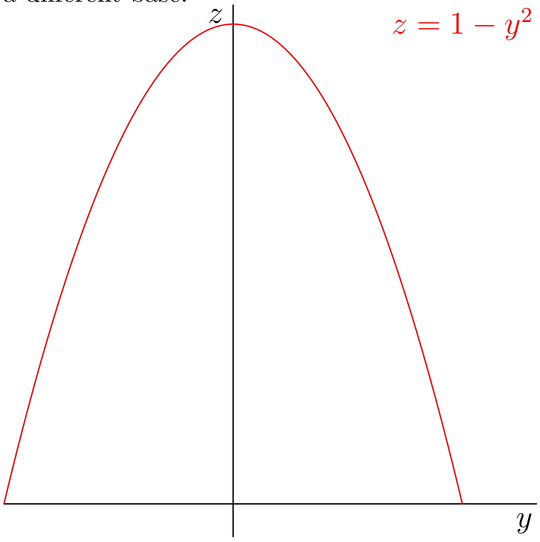
We can now see that

$$\iiint_V \mathrm{d}V = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}z \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \mathrm{d}y$$

which trivially equals

$$\int_0^1 \mathrm{d}z \int_0^{1-z} \mathrm{d}x \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \mathrm{d}y.$$

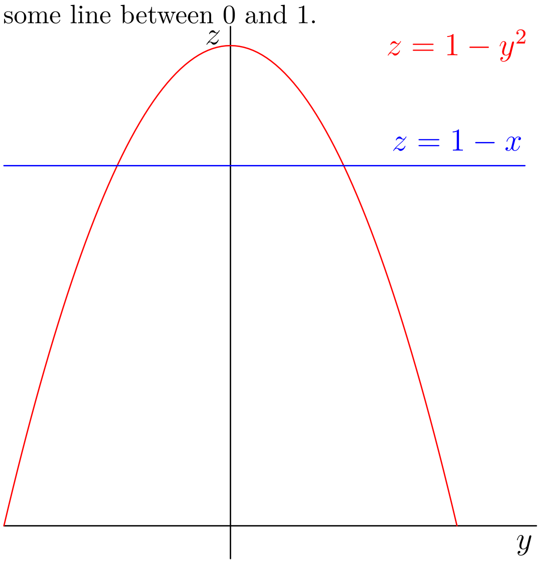
As you can see, switching the outer two integrals is no big deal since we are informed by our base. But, let us consider a different base.



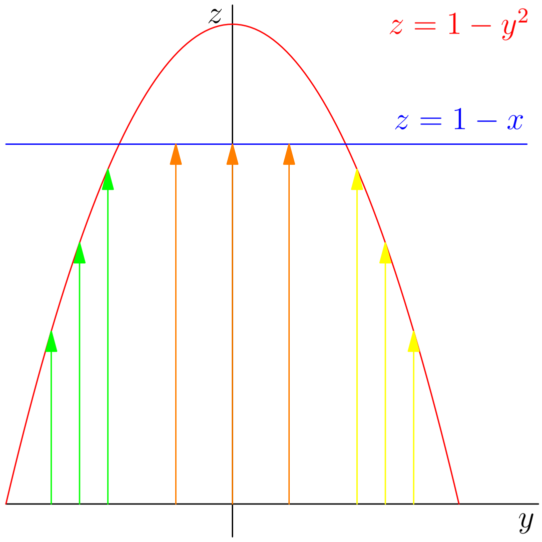
We can now see that if we we send y out first from $-1 \to 1$, z with go from $0 \to 1 - y^2$. Then use the final equation to relate x and z to receive

$$\int_{-1}^1 \mathrm{d}y \int_0^{1-y^2} \mathrm{d}z \int_0^{1-z} \mathrm{d}x = \int_0^1 \mathrm{d}z \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \mathrm{d}y \int_0^{1-z} \mathrm{d}x.$$

Again, swapping the two outer integrals is easy. There are only two integrals left, both of which do not have a clear base (it is as if we try to take xy plane as the base). We will do an inner swap to try to achieve this. Note that we cannot write a single integral for these two. Recall the yz graph. Our newest integral order would be xyz. So x is held constant. Given $z = 1 - x$, where is this graphed? Well, since $0 < x < 1$, $0 < 1 - x < 1$. Therefore, $z = 1 - x$ is some line between 0 and 1.



Now, we said we would send out y next, which goes $0 \to 1$. Where does z go? It sometimes goes to $z = 1 - y^2$. But sometimes, it goes to the line $z = 1 - x$. It simply depends where it is. The intersection point is $1 - y^2 = 1 - x \implies y = \pm x$. Therefore, if $y < -\sqrt{x}$, then z goes to $1 - y^2$ (green arrows). If $-\sqrt{x} < y < \sqrt{x}$, then z goes to $1 - x$ (orange arrows). Otherwise, $1 - y^2$ (yellow arrows).



The integral is

$$\int_0^1 \mathrm{d}x \int_{-1}^{-\sqrt{x}} \mathrm{d}y \int_0^{1-y^2} \mathrm{d}z + \int_0^1 \mathrm{d}x \int_{-\sqrt{x}}^{\sqrt{x}} \mathrm{d}y \int_0^{1-x} \mathrm{d}z + \int_0^1 \mathrm{d}x \int_{\sqrt{x}}^{1} \mathrm{d}y \int_0^{1-y^2} \mathrm{d}z$$

Without coloring that is

$$\int_0^1 \mathrm{d}x \int_{-1}^{-\sqrt{x}} \mathrm{d}y \int_0^{1-y^2} \mathrm{d}z + \int_0^1 \mathrm{d}x \int_{-\sqrt{x}}^{\sqrt{x}} \mathrm{d}y \int_0^{1-x} \mathrm{d}z + \int_0^1 \mathrm{d}x \int_{\sqrt{x}}^{1} \mathrm{d}y \int_0^{1-y^2} \mathrm{d}z.$$

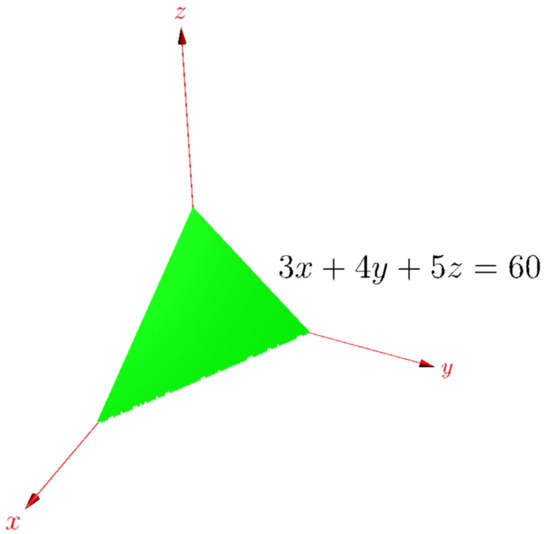
Performing an outerswap gives the last integral. □

4.1 Tetrahedrons

Example. Given V is the region in the first octant bounded by $3x + 4y + 5z = 60$, determine the value of

$$\iiint_V x^2 yz \, \mathrm{d}V.$$

Solution. There are two ways to do this. But first, let us sketch the region.



Let us send x out first. We receive

$$\int_0^{20} \mathrm{d}x \int_0^{\frac{60-3x}{4}} \mathrm{d}y \int_0^{\frac{60-3x-4y}{5}} \mathrm{d}z \cdot (x^2 yz).$$

Solving this integral as is would give you and your professor a nightmare. Let us do some u-substitutions instead! As previously mentioned, there are two ways to do this. Let us examine the first substitution.

$$u = \frac{x}{20} \to x = 20u$$
$$v = \frac{y}{15} \to y = 15v$$
$$w = \frac{z}{12} \to z = 12w$$

Into the integral it goes!

$$\int_0^1 20\mathrm{d}u \int_0^{1-u} 15\mathrm{d}v \int_0^{1-u-v} 12\mathrm{d}w \cdot (20^2u^2 15v 12w).$$

Not bad, but solving this integral will still give you a minor headache. We can come up with a better substitution!

$$\begin{aligned}
u &= \frac{x}{20} \to x = 20u \\
v &= \frac{y}{\frac{60-3x}{4}} \to y = \frac{60-3x}{4}v = \frac{60-60u}{4}v = 15(1-u)v \\
w &= \frac{z}{\frac{60-3x-4y}{5}} \to z = \frac{60-60u-60(1-u)v}{5}w = 12(1-u-(1-u)v)w = 12(1-u)(1-v)w.
\end{aligned}$$

Then, integrate.

$$\begin{aligned}
\iiint_V x^2yz \ \mathrm{d}V &= \int_0^1 20\mathrm{d}u \int_0^1 15(1-u)\mathrm{d}v \int_0^1 12(1-u)(1-v)\mathrm{d}w \cdot (20^2u^2 \cdot 15(1-u)v \cdot 12(1-u)(1-v)w) \\
&= 20^3 15^2 12^2 \int_0^1 (1-u)^4u^2\mathrm{d}u \int_0^1 v(1-v)^2\mathrm{d}v \int_0^1 w\mathrm{d}w
\end{aligned}$$

A nice factored integral, isn't it? From here, we will quickly do another few substitutions. Consider $s = 1-u$. Then $\mathrm{d}s = -\mathrm{d}u$. This means the bounds on integration will switch. But, the new bounds via plugging into s are also switched. So it switches twice, which is the same as not switching at all. Think of it this way:

$$\int_0^1 (1-u)\mathrm{d}u = \int_{1-0}^{1-1} s \cdot -\mathrm{d}s = \int_{1-1}^{1-0} s \ \mathrm{d}s = \int_0^1 s \ \mathrm{d}s.$$

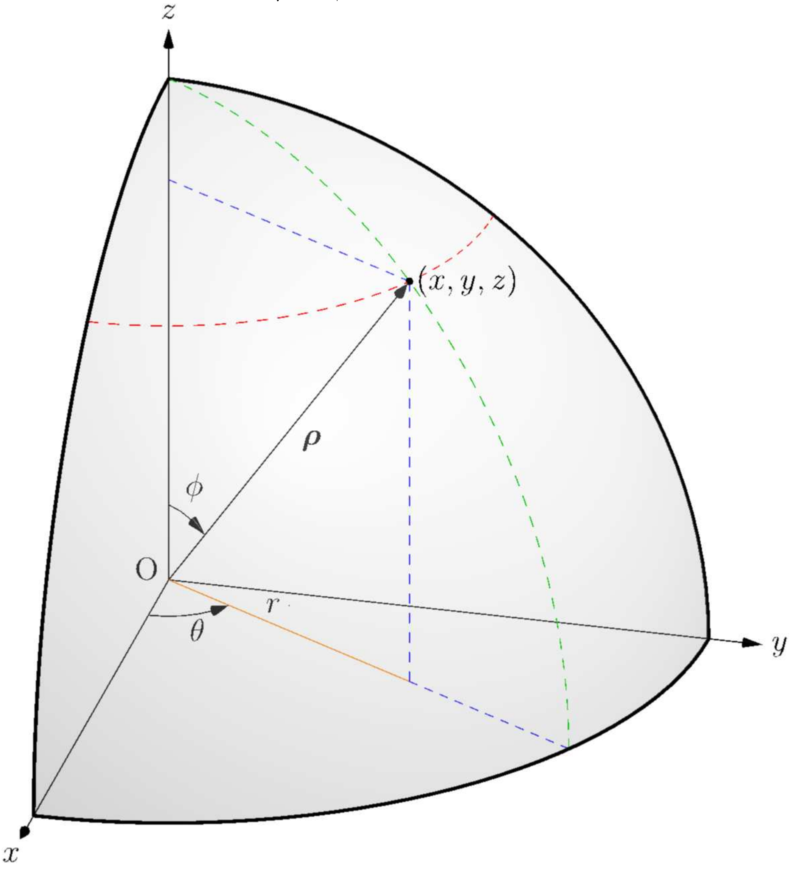
So completing the above integrals are trivial and left as an exercise. $\boxed{\frac{2^7 3^2 5^4}{7}}$ □

4.2 Cylindrical and Spherical

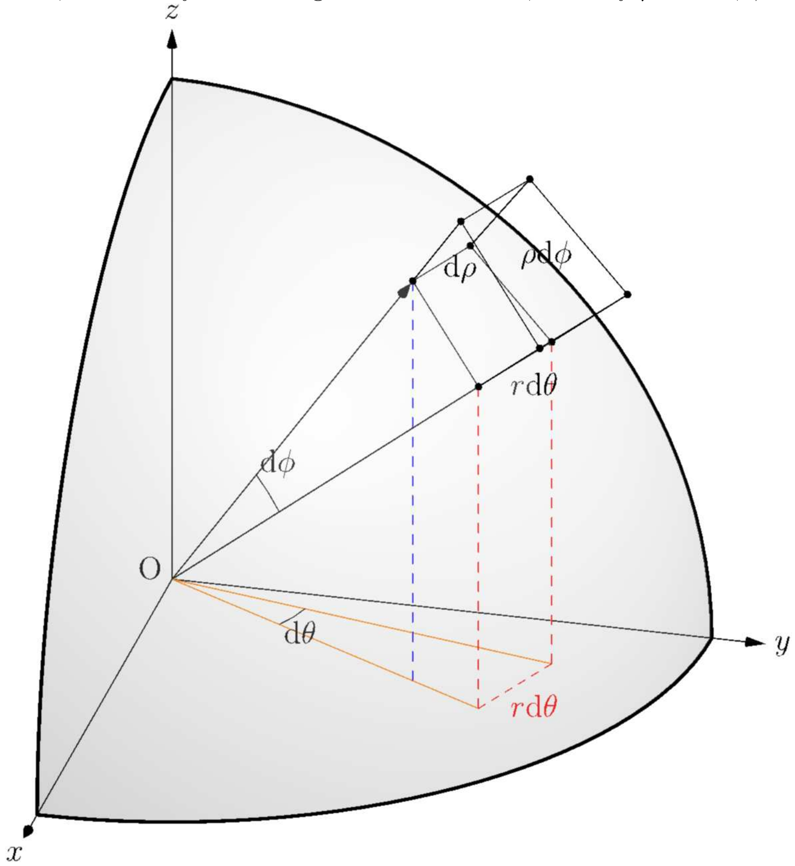
The cylindrical substitution is the exact same as polar. The z integral is left untouched and xy switches to polar. Spherical is somewhat different. Let us do some more differential analysis.

Theorem 4.1. $\mathrm{d}V = \rho^2 \mathrm{d}\rho \sin\phi \ \mathrm{d}\phi \mathrm{d}\theta$.

Proof. Recall that $r = \rho\sin\phi$. Observe:



Now, let us vary each thing a little bit. First, we vary ρ. Then, ϕ. Then, θ.



The side lengths of the spherical rectangular prism are: dρ, ρdϕ (by arclength), and $r\mathrm{d}\theta = \rho\sin\phi\mathrm{d}\theta$ (by arclength). □

Example. Find the average value of $2x^3y^2 + 3x^2 + 8xy^4z^2 + 2x^2y^2$ on the ball $x^2 + y^2 + z^2 \leq 8$.

Solution. Realize that we have symmetry in x, y, and z. Now, let us calculate the numerator integral.

$$\iiint_V (2x^3y^2 + 3x^2 + 8xy^4z^2 + 2x^2y^2)\mathrm{d}V = \iiint_V (\cancelto{0}{2x^3y^2} + 3x^2 + \cancelto{0}{8xy^4z^2} + 2x^2y^2)\mathrm{d}V.$$

Now, consider this. There is symmetry in all the directions. Therefore, $\iiint_V x^2\mathrm{d}V = \iiint_V z^2\mathrm{d}V$. That is how spheres work. You can't functionally distinguish between x and z in a sphere. Or y and z. So you can even transform x^2y^2 to x^2z^2. You CANNOT transform x^4 to something like x^2z^2. Remember, $\int\sum = \sum\int$ but $\int\prod \neq \prod\int$ in general. Use this to decide if your switching is well motivated. We want to change to z since it is easier in spherical coordinates. After all, $y = \rho\sin\phi\sin\theta$ while z just equals $\rho\cos\phi$.

$$\iiint_V (3x^2 + 2x^2y^2)\mathrm{d}V = \iiint_V (3z^2 + 2z^2y^2)\mathrm{d}V.$$

At this point, we switch to spherical.

$$\begin{aligned}
&\iiint_V (3\rho^2\cos^2\phi + 2\rho^2\cos^2\phi\rho^2\sin^2\phi\sin^2\theta)\mathrm{d}V \\
&= \int_0^{2\sqrt{2}} \rho^2 \ \mathrm{d}\rho \int_0^{\pi} \sin\phi \ \mathrm{d}\phi \int_0^{2\pi} \ \mathrm{d}\theta \cdot (\text{integrand}) \\
&= \int_0^{2\sqrt{2}} \rho^2 \ \mathrm{d}\rho \int_0^{\pi} \sin\phi \ \mathrm{d}\phi \cdot \left(6\pi\rho^2\cos^2\phi + 2\pi\rho^4\cos^2\phi\sin^2\phi\right) \\
&= 2\pi \int_0^{2\sqrt{2}} \rho^4 \ \mathrm{d}\rho \int_0^{\pi} \sin\phi \ \mathrm{d}\phi \cdot \left(3\cos^2\phi + \rho^2\cos^2\phi\sin^2\phi\right)
\end{aligned}$$

We will make a very common substitution which is $u = \cos\phi$. This substitution will come up a lot during these integrals and some professionals consider it standard. Note that $\mathrm{d}u = -\sin\phi \ \mathrm{d}\phi$ which means we have to switch bounds.

$$
\begin{aligned}
&2\pi \int_0^{2\sqrt{2}} \rho^4 \, \mathrm{d}\rho \int_{-1}^{1} \mathrm{d}u \cdot \left(3u^2 + \rho^2 u^2 (1-u^2)\right) \\
&= 4\pi \int_0^{2\sqrt{2}} \rho^4 \, \mathrm{d}\rho \int_{0}^{1} \mathrm{d}u \cdot \left(3u^2 + \rho^2 u^2 (1-u^2)\right) \\
&= 4\pi \int_0^{2\sqrt{2}} \rho^4 \left(1 + \frac{\rho^2}{3} - \frac{\rho^2}{5}\right) \mathrm{d}\rho \\
&= 4\pi \int_0^{1} \left(2\sqrt{2}\right)^4 v^4 \left(1 + \frac{2\cdot\left(2\sqrt{2}\right)^2 v^2}{3\cdot 5}\right) \left(2\sqrt{2}\right) \mathrm{d}v \\
&= \left(2\sqrt{2}\right)^5 4\pi \left(\frac{1}{5} + \frac{2^4}{3\cdot 5\cdot 7}\right) \\
&= \frac{2^9 37\sqrt{2}\pi}{3\cdot 5\cdot 7}.
\end{aligned}
$$

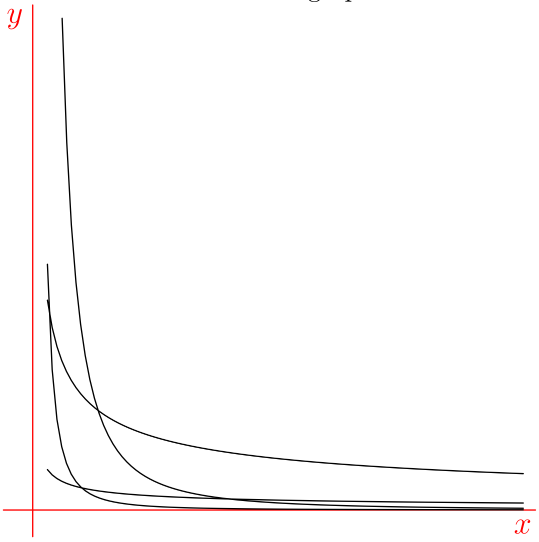
The denominator is the volume of the ball, which can either be computed via an integral or simply realized to be $\frac{2^6\sqrt{2}\pi}{3}$. The average value is $\boxed{\frac{2^3 37}{5\cdot 7}}$. □

5 The Jacobian

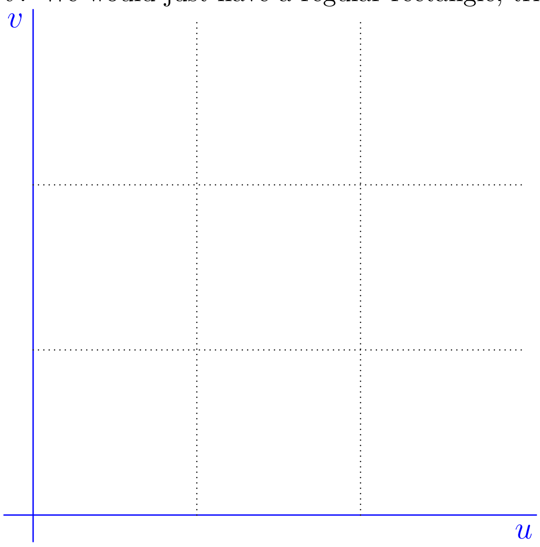
We would like to generalize the notion of u-substitution into higher dimensions. We have already been doing this by applying polar coordinates. However, what if we would like to make different substitutions?

Example. Find the average value ofx^2y on the region bounded by $xy^2 = 1$, $xy^2 = 8$, $x^2y = 1$, $x^2y = 27$.

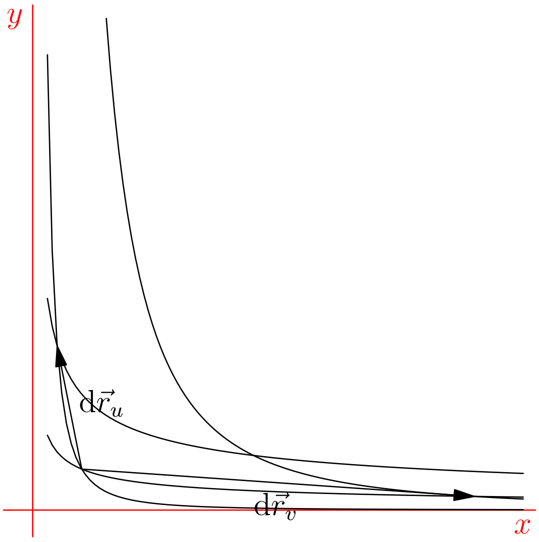
Solution. Let us sketch a graph.



As seen, this will be extremely ugly. Let $u = xy^2$ and $v = x^2y$. What if we lived in a world where we did u against v? We would just have a regular rectangle, trivial!



So we will seek to convert it to that. There are three things to change about our integral. First, the limits of integration. This is trivial as we are now integrating over a rectangle. The integrand itself is also easy to change, we have the information we need. The differential is interesting.



$d\vec{r}_u$ means change in u along constant v. In the graph, we are finding change in u for $v = 1$. $d\vec{r}_v$ is the other way. Another way to think about this is $d\vec{r}_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \rangle du$. Notably, these form a parallelogram, NOT a rectangle. We know the area of a parallelogram is $dA = |d\vec{r}_u \times d\vec{r}_v|$. Let us compute:

$$\begin{aligned} |d\vec{r}_u \times d\vec{r}_v| &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} du\ dv \\ &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\ dv \end{aligned}$$

This is written as $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. Now, after solving for x and y in our original equations, we get $x = u^{-\frac{1}{3}}v^{\frac{2}{3}}$ and $y = u^{\frac{2}{3}}v^{-\frac{1}{3}}$. We could calculate the partial derivatives this way, which would just be a lot of work. I will offer an alternative approach. Suppose we convert $dxdy$ to $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$. Now, what if we convert $dudv$ back using the same idea? That results in $dudv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy$. Combining, we get $dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy$. Thus,

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}^{-1}.$$

Calculating $\frac{\partial u}{\partial x}$ is much easier. In our case, after calculating the determinant, we get $|-3x^2y^2|^{-1} = \frac{1}{3x^2y^2} = \frac{1}{3}u^{-\frac{2}{3}}v^{-\frac{2}{3}}$. Our integral setup is as so:

$$\int_1^8 du \int_1^{27} dv \left(\frac{1}{3}u^{-\frac{2}{3}}v^{-\frac{2}{3}}\right) v.$$

This is a trivial integral that does not involve anything special, so it is left as an exercise. The denominator is also left as an exercise. The answer is $\boxed{10}$. □

You can also make further substitutions (i.e., Jacobian then into polar) which is perfectly fine.

⚠ In general, the transformed Jacobian version does not play nicely with the old coordinate system. For example, you CANNOT find the centroid in (u, v) and then convert back to (x, y). You have to be considering (x, y) from the start.

Exercise. Use the Jacobian to show that the Polar differential is $r dr d\theta$.